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COMPILED BY MISS A. J. STEAL.

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PROJECTIVE GEOMETRY.

PROJECTIVE geometry derives both interest and importance from the fact that it assumes only the simplest and most fundamental axioms. The following sketch of the first principles, and proof of the fundamental theorem, is taken from Prof Federigo Enriques' *Projektive Geometrie* (B. G. Teubner, Leipzig, 1903, 8vo, pp. xiv, 374). This work is specially adapted for school teaching. The first (Italian) edition appeared in 1898, and the axioms on which it is founded were explained four years earlier in the *Rendiconti R. Ist. Lombardo*, Vol. 27, pp. 550-567.

Projective geometry is the geometry originated by von Staudt in the *Geometrie der Lage*, of which an excellent account by Prof. C. A. Scott has already been given in the *Mathematical Gazette*, Vol. I., Nos. 19, 20, 22, pp. 307, 323, 363. Russell's very original *Foundations of Geometry* is the only English work which treats the subject from a strictly logical standpoint, but is unfortunately very unreliable as far as the mathematics is concerned. The more serious errors are tacitly abandoned in his recent book on *The Principles of Mathematics*; but a large part of this important work is very indefinite, or at least leaves no definite impressions; and its use of technical terms, without any guiding reference, is very perplexing.

Opinions differ as to how projective geometry should be defined, and to what extent its axioms should be restricted. It is usual to regard it as including all those properties and relations of figures which can be stated without reference to the conception of magnitude. There are different varieties (elliptic, parabolic, hyperbolic), but they differ much less from one another than the corresponding types of metrical geometry. Two notions seem to be essential (1) that space is divisible into elements, called points, certain derived elements being called straight lines and planes, and (2) that these elements can be arranged systematically, so that *order* (or a co-ordinate system) results from the arrangement Order must be given either explicitly or implicitly by the axioms. A further axiom, the axiom of continuity, is essential for the development of projective geometry as at present understood, since otherwise only a very mutilated theory of double points would be possible. It has not been proved, however, that no theory of projective geometry is possible without this axiom. The axiom is also open to the objection that it appears to be based too intimately on preconceived notions of arithmetic.

Two straight lines in a plane which do not meet in a proper point are said to have an improper or ideal point common. Ideal points can be made to correspond projectively to proper points. We shall regard all imaginaries as excluded, owing to the unsatisfactory way in which they must be introduced, and the complicated considerations to which they give rise.

Enriques, following von Staudt, regards projective geometry as visual and intuitional; and assumes the *axioms of Euclid with the exception of* the axiom of superposition. In place of saying that two lines are parallel (which in Euclid involves a metrical notion) he says that they have an improper or ideal point common. Thus every straight line contains one and only one ideal point, which has the effect of *closing* the line. Two points A, B on a line divide it into two segments; and if C is in one segment, and D in the other, the pairs A, B and C, D are said to separate one another. A plane contains *one* ideal line (we omit the prefix *straight*), and space contains *one* ideal plane. The elements of space, viz. points, lines, and planes, are given intuitionally. In space of three dimensions either points or planes may be regarded as the original elements; but in both cases lines are derived elements. Forms of three grades are distinguished; in the first grade are the series or range of all points on a line, the flat pencil of all lines in a plane through a point, and the axial pencil of all planes through a line, the point and line being called the vertex and axis of the two pencils respectively. Forms of the second grade include the point-system of a plane, the line-system of a plane, *the sheaf of all lines through a point*, and the sheaf of all planes through a point. The forms of the third grade are the point-system of space and the plane-system of space. The line-system of space is of a higher grade, giving rise to line geometry. Two forms are perspective if both are sections of the same form, or if one is a section of the other, such as an axial pencil and its section by a line or plane. Two forms are projective if it is possible to pass from one to the other by means of a finite number of perspectivities. For a full discussion of these conceptions and definitions see Prof. Scott's articles in the *Mathematical Gazette* referred to above.

Enriques divides the axioms into two sets of three each. The first set presents no distinctive peculiarity, and states no property which has not been generally admitted as axiomatic. The first set of axioms is as follows:—

I. In a form of the third grade two original elements determine a like form of the first grade, to which the two elements belong.

It is of course understood that the form determined by the two elements is contained in the given form of the third grade. A similar remark applies to axioms II. and III. The elements include *all* the elements, both proper and ideal.

Thus (a) two points determine *one* straight line, or range of points, to which they belong; and (b) two planes determine *one* straight line, or axial pencil, to which they belong.

II. In a form of the third grade three original elements, which do not belong to a form of the first grade, determine a like form of the second grade, to which the three elements belong.

Thus (c) three points, not on a straight line, determine (the point-system of) *one* plane; and (d) three planes, not through a common line, determine *one* sheaf of planes.

III. In a form of the third grade an original element and a form of the first grade, which do not belong to one another, determine a form of the second grade, to which they belong.

Thus (e) a point and range determine one plane; and (f) a plane and axial pencil determine *one* sheaf of planes.

From these axioms it also follows that in a form of the second grade two forms of the first grade have always *one* element common. Also von Staudt's theorem concerning harmonic sets of four elements follows without any further axioms. Before considering this, however, it will be more convenient to give Enriques' remaining axioms. These are more difficult to decide upon, and also more debatable, than the first set of axioms given above. The principal part of the fourth axiom is as follows:—

IV. The elements of a form of the first grade possess a natural cyclical order, which may be taken in the one or the opposite sense. Any element may be taken as the first element; but between any two elements there are an infinite number of other elements.

Four elements can be divided into two pairs which separate one another in only one way. Thus if they occur in the order A, B, C, D then A and C separate B and D. Also two elements divide a form of the first grade into two parts or segments.

V. If two like forms of the first grade are perspective and an element describes a segment AB of the one, the corresponding element describes a segment $A'B'$ of the other, where A' corresponds to A and B' to B.

To every element X in one form corresponds *one* element X' in the other, by the first set of axioms; and so long as X lies in one particular segment of the first form, axiom V. states that X' lies in a particular segment of the other, and when X goes over to the other segment so does X' go over to the other.

Axioms IV. and V. are capable of analysis, and can be reduced to others which are not so comprehensive: but, regarded as intuitive axioms, nothing could be simpler, or better adapted for elementary teaching. There remains only the axiom of continuity, or Dedekind's axiom.

VI. If a segment AB of a form of the first grade is divided into two parts, so that—

(1) each element of the segment AB belongs to one part or the other,

(2) A belongs to the first part and B to the second,

(3) each element of the first part is before each element of the second part; then there exists *one* element C of the segment AB (which may belong to either part) such that each element of AB which goes before C belongs to the first part, and each element of AB which comes after C belongs to the second part.

We now come to the well-known theorem, which von Staudt showed to be a consequence of the first set of axioms, that in a form of the first grade to three given elements corresponds a fourth unique element forming with the three a harmonic set. It will be sufficient to prove this for the range of points on a line. Let E, H, F be the three given points (see figure next page). On a line through H take any two points A, C; let EA, FC meet in B, and EC, FA in D; then the point K where BD meets EF is the fourth unique element. H, K are called a harmonic pair with respect to E, F, and may be described as the points in which EF, regarded as a diagonal of any quadrilateral $ABCD$, is cut by the other two diagonals.

To prove that K is unique, draw any other line $HA'C'$ through H, not lying in the same plane as HAC and EF, and construct the quadrilateral $A'B'C'D'$ and the point K' corresponding to H, A', C'. Then AA' and BB' lie in the plane containing the two lines EAB and $EA'B'$; hence AA' and BB' meet one another, similarly BB' and CC' meet, and CC' and AA' meet; therefore AA', BB', CC' meet in one and the same point, for they do not lie in one plane, since the planes ABC, $A'B'C'$ are different by hypothesis.

By a similar proof DD' passes through the point where AA' and CC' meet, and therefore meets BB'; hence BD and $B'D'$ meet, and both meet EF (in K and K'); therefore BD, $B'D'$, EF, not lying in one plane, meet in one and the same point, *i.e.* K' coincides with K.

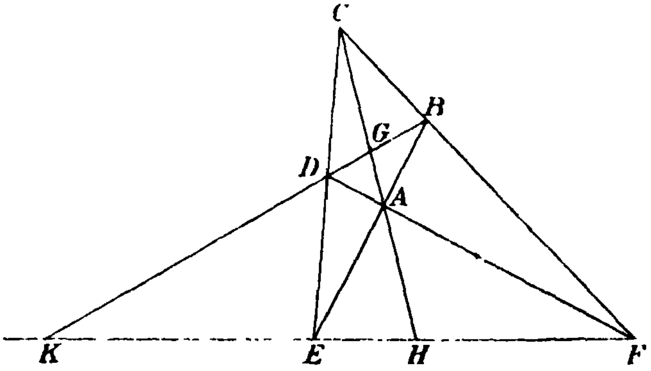
Also, if we construct the point K'' for any quadrilateral in the same plane as $ABCD$, having EF for a diagonal, then K'' coincides with K', and therefore with K; hence K is unique, Q.E.D.

The above proof would fail if we were restricted to geometry of two dimensions. In this case either points or straight lines could be regarded as original or fundamental elements; and there would be only one axiom corresponding to those of the first set above, viz.:—In a form of the second grade two original elements determine a form of the first grade, to which they belong. It would seem that this axiom would have to be supplemented by others in order to prove the uniqueness of the fourth element of a harmonic set. In two dimensions there is no set of elements, like straight lines in three dimensions, which cannot be regarded as original elements.

In two perspective ranges a harmonic set always corresponds to a harmonic set; and the same is therefore true for any two projective ranges. This is seen by constructing any quadrilateral for one harmonic set, and considering the corresponding perspective quadrilateral.

If H, K is a harmonic pair with respect to E, F it is clear that the relation between H, K is a reciprocal one; so also is the relation between the two pairs H, K and E, F, *i.e.* E, F is a harmonic pair with respect to H, K. In the figure let FG meet AB in A' and CD in C'', and let EG meet BC in B' and DA in D'. From the quadrilateral $GB'CC''$ we see that $B'C''$ passes through K; and similarly from the quadrilaterals $GD'AA'$, $GA'BB'$, $GC''DD'$ we see that $D'A'$ passes through K, and $A'B'$ and $C''D'$ pass through H. Hence HK is a diagonal of the quadrilateral $A'B'C''D'$, and the other two diagonals meet HK in E, F, which proves the theorem. The triangle GHK formed by the diagonals of the quadrilateral $ABCD$ is its harmonic triangle, and is also the harmonic triangle of the quadrangle $A'B'C''D'$; EFG is the harmonic triangle of the quadrilateral $A'B'C''D'$ and of the quadrangle $ABCD$.

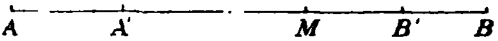
The next important theorem which we have to consider is the following:—Any harmonic pair separates the pair with respect to which it is harmonic, *i.e.*, the harmonic pairs of any harmonic set are the two pairs which separate one another. This is proved by means of axioms IV. and V.



In the figure we have to prove that H, K separate E, F. Take a point P in EAB such that E, P, A, B occur in order; let DP cut HAC in O and EF in X, and let BO cut ECD in Q and EF in Y. Then projecting E, P, A, B from D on to EF, we see that E, X, F, K occur in order, or X lies in the opposite segment EF to that in which K lies. Projecting E, P, A, B form O on to ECD, we have E, D, C, Q, and projecting these from B on to EF, we have E, K, F, Y; therefore K, Y lie in opposite segments EF, *i.e.* X, Y lie in the same segment EF, and K in the opposite one. Finally project E, P, A, B from O on to EF, and we have E, X, H, Y. Hence in considering the order E, X, F we have E, X, H, Y, F or E, Y, H, X, F in order, according as X is before or after Y; *i.e.* H is in the segment EF which contains X, Y, and does not contain K, Q.E.D.

The *fundamental theorem* is the following:—In two projective forms of the first grade any three elements of one may correspond to any three elements of the other; and then to any fourth element of one corresponds a unique element of the other. We shall prove this as before for ranges of points only.

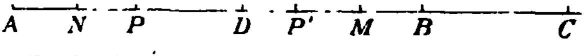
(1) Let A, A', B', B be four points in order on a line, and let AB, $A'B'$ be corresponding segments of two projective ranges, so that, as a point describes the segment AB in the sense $AA'B$ the corresponding point describes $A'B'$ *in the same sense.* Then there exists a double point M within the segment $A'B'$, such that there is no double point in $AA'M$ before M.



The segment AB can be divided into two parts, of which the first part contains all points X possessing the property that each and every point in the segment AX goes before its corresponding point, and the second part contains all points Y possessing the property that some point exists in AY (which may be Y itself) which does not go before its corresponding point. The points X, Y make up all the points of the segment A, B; also A is a point X and B is a point Y; also every X goes before every Y; hence (axiom VI.) there is a point M in the segment $AA'B$ such that every point before M is an X, and every point after M is a Y. Also, since the correspondent of any point in AB is in $A'B'$, it is clear that every point in AA' (excluding A') is an X, and every point in $B'B$ a Y. Hence M is somewhere in $A'B'$.

Let M' be the point which corresponds to M. Then, if M' is before M, any point within $M'M$ is an X, whose correspondent X' is after X, and therefore after M'. But since X is before M, X' is before M' (axiom V.); therefore X' is both before and after M', which is impossible. Again, if M' is after M, any point within MM' is a Y, whose correspondent Y' is after M' (because Y is after M), and therefore after Y. Thus Y is an X, by definition, which is impossible. Hence M' is neither before nor after M, and must coincide with M, *i.e.* M is a double point. Also, there is no double point in $AA'M$ before M, since every point before M is an X.

(2) If two projective ranges on the same line have three double points, then every point is a double point, *i.e.* the projectivity is identical (fundamental theorem of von Staudt).



Let A, B, C be the three double points, and suppose that the projectivity is not identical, so that there is some point P whose correspondent P' does not coincide with P.

We may assume that P does not lie in the segment ACB; for if A', B', C' are three points such that A, A' separate B, C and B, B' separate C, A and C, C' separate A, B, then A, C', B, A', C, B' occur in order; and P must lie in one of the segments $AC'B$, $BA'C$, $CB'A$, and we may suppose it to lie in $AC'B$. The segment ACB corresponds to itself, since C corresponds to itself; hence APB also corresponds to itself, and therefore P' lies in APB. We may also assume that A, P, P', B occur in order, by interchanging A and B if they are not in order.

The segment $P'B$ corresponds to PB, both taken in the sense APB. Hence, by (1), there is a double point M in $P'B$, which may be B, such that there is no other double point in PM. The reasoning of (1) still holds when the corresponding points B B' coincide, as is the case here. Again, reversing the order and considering the reciprocal projectivity, to the segment $P'A$ corresponds the segment PA, both in the sense BPA. Hence there is a double point N in PA, which may be A, such that no other double point exists in $P'N$. We have thus found two double points N, M, such that no other double point exists in the segment NPM.

Take the point D such that C, D and M, N are harmonic pairs. Then since any set corresponding to a harmonic set is harmonic, and each of the points C, M, N corresponds to itself, D can only correspond to itself. Hence D is a double point within the segment NPM, which is impossible from above. Hence a projectivity which contains three double points must be identical.

(3) In two projective ranges any three points of one may correspond to any three points of the other.

When the ranges are on two lines not in the same plane any three points of one A, B, C are perspective with any three points A', B', C' of the other. Take any point L on AA', other than A or A', and let the plane LCC' meet BB' in M; then LM meets CC' in a point N. Then considering the axial pencil with LMN as axis, it is clear that the three planes of the pencil which meet ABC in A, B, C will meet $A'B'C'$ in A', B', C' respectively; *i.e.* A, B, C are perspective with A', B', C'.

If the lines are the same, or in the same plane, take any three points A'', B'', C'' on a line which is not in a plane with either ABC or $A'B'C'$. Then A, B, C and A', B', C' are both perspective with A'', B'', C'', and therefore projective with one another.

Finally if A, B, C, D on one line are projective with A', B', C', D' on another (or the same), and A, B, C, D, A', B', C' are fixed, then D' is fixed; for if A, B, C, D are also projective with A', B', C', D'', then A', B', C', D' are projective with A', B', C' D'', and therefore D'' coincides with D', by (2). This completes the proof of the fundamental theorem for projective ranges; and, in the same way, the theorem may be proved for any two projective forms of the first grade.

We have only referred to a small portion of Enriques' *Projektive Geometrie*, but we have given sufficient to show that it is a book of more than ordinary interest. The whole work is a very complete one, and includes the theory of conics and of projectivity in forms of all the three grades. In another article we may consider the projective coordinate system, which does not come within the scope of Enriques' book.

F. S. MACAULAY.

MATHEMATICAL NOTES.

142. [K. 20. d.]. *To prove geometrically the principal trigonometrical relations of two angles.*

Let O be the centre, OX a radius of a circle whose diameter $= 1$.

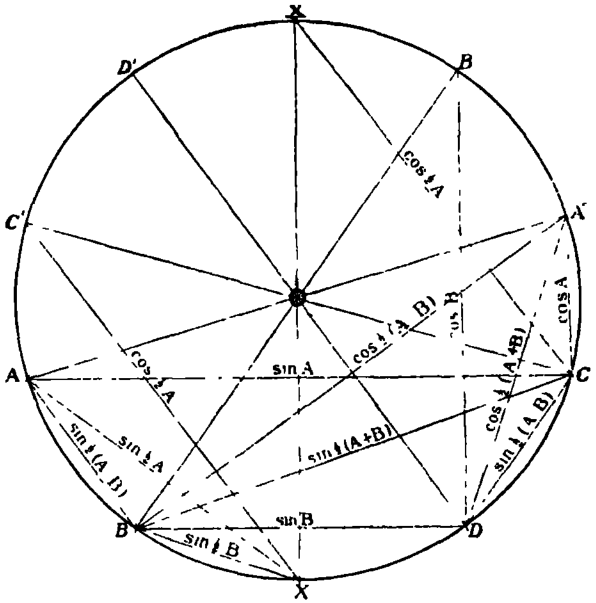
Draw the radii OA, OC making angle A on each side of OX.

Similarly the radii OB, OD making angle B.

Let X', A', B', etc. be the extremities of diameters from X, A, B, etc.

Then we can write down

$$\begin{array}{r|r}
\sin A = AC = A'C'. & \cos A = AC' = A'C. \\
\sin B = BD = B'D'. & \cos B = BD = B'D. \\
\sin\frac{A}{2} = AX = CX. & \cos\frac{A}{2} = AX' = C'X. \\
\sin\frac{B}{2} = DX = BX. & \cos\frac{B}{2} = BX' = DX'. \\
\sin\frac{A+B}{2} = AD = BC. & \cos\frac{A+B}{2} = AD = BC'. \\
\sin\frac{A-B}{2} = AB = CD. & \cos\frac{A-B}{2} = A'B = CD'.
\end{array}$$



Now apply Euclid VI. D. to different quadrilaterals in turn, and we write down the required relations at sight:

Quadrilateral	*gives*
$AXCX'$	$\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$.
$BXCX'$	$\sin\frac{A+B}{2} = \sin\frac{A}{2}\cos\frac{B}{2} + \cos\frac{A}{2}\sin\frac{B}{2}$.
$ABXX'$	$\sin\frac{A-B}{2} = \sin\frac{A}{2}\cos\frac{B}{2} - \cos\frac{A}{2}\sin\frac{B}{2}$.
$ACBD$	$\sin A\sin B = \sin^2\frac{A+B}{2} - \sin^2\frac{A-B}{2}$.
$ABCB'$	$\sin A = \sin\frac{A+B}{2}\cos\frac{A-B}{2} + \cos\frac{A+B}{2}\sin\frac{A-B}{2}$.
$AA'BD$	$\sin B = \sin\frac{A+B}{2}\cos\frac{A-B}{2} - \cos\frac{A+B}{2}\sin\frac{A-B}{2}$.
$ADCB'$	$\sin A\cos B = \sin\frac{A+B}{2}\cos\frac{A+B}{2} + \sin\frac{A-B}{2}\cos\frac{A-B}{2}$.

and so on.

W. A. WHITWORTH.

107. [P. 3. B.] *Continued Inversion by Coaxal Circles.*

Mr. R. F. Davis's Note (p. 383) on Double Inversion suggests some curious consequences. Remarking that his three circles of inversion (A), (B), (E), are all orthogonal to PQR, and are therefore coaxal; and that the bisector of EF, being a diameter of PQR, is their radical axis; we may state his theorem thus:

Successive inversion by any two circles $(A)(B)$ is equivalent to inversion by the coaxal circle (E) followed by reflection in the radical axis (R); or to reflection in (R) followed by inversion by the coaxal circle (F).

Or briefly, in symbols,

$$[AB]=[ER]=[RF].$$

The points B, E are inverse for (A), and A, F are inverse for (B). When (A), (B) are orthogonal, E and F coincide, and $[AB]=[BA]$.

(1) Take (C) any other circle coaxal with (A), (B), and let F, D be inverse points for (C). Then

$$[ABC]=[RFC]=[RRD]=[D].$$

And evidently we can carry on this process to any extent, till we get the result:

Successive inversion by any odd number $(A)(B)(C)\ldots(K)$ of a series of coaxal circles amounts only to a simple inversion by that one of the series whose centre is where the point A arrives after inversion by $(B)(C)\ldots(K)$.

Another way of finding this centre would be to perform the operation $[ABC\ldots K]$ on a point P, obtaining Q, R, … P', (which are all concyclic), and to draw PP'.

(2) $[ABCABC]=[DD]=1$; that is, the six successive inversions produce no change.

(3) Let (C) be the circle into which (B) inverts (A). Then, choosing P on the circumference of (A), we find that D coincides with B; $\therefore$ $[ABC]=[B]$. This result includes two others, well known: (a) when a circle (A) is inverted into another (C) the inverse of the centre of (A) is with respect to (C) the inverse of the centre of inversion: and (b) when a circle (A) and two points inverse for (A) are inverted, they become a circle (C) and two points inverse for (C).

(4) Let $(A)(B)(C)$ be a set of circles each of which inverts the other two into each other. Then, by (3),

$$[AB]=[AABC]=[BC]$$

Similarly $[BC]=[CA]$; and $[AC]=[CB]=[BA]$. Hence:

If any point P be taken, and its three inverses for $(A)(B)(C)$, and of these again all the possible inverses, and so on to any extent, no more than five new points will be obtained; namely, those which result from the operations $[A]$, $[B]$, $[C]$, $[AB]$, $[AC]$; for all other operations, however long and varied, can be reduced to one of these.

These circles $(A)(B)(C)$, if real, must have common points O, O'; otherwise each circumference would have to pass between the other two. And of the radii OA, OB, OC, each bisects the angle between the other two, because for instance $AB:AC=OB:OC$; hence the angles of intersection are all 60° or 120°. The three Apollonian circles of a triangle have this property.

C. E. YOUNGMAN.

REVIEWS.

The Algebra of Invariants. By J. H. GRACE, M.A., and A. YOUNG, M.A.

The Theory of Invariants, since it was commenced by Boole in 1844, has shewn, now for nearly sixty years, a continuous development. It has given rise to several entirely new methods, which have in succession enabled fresh discoveries to be made. When, for example, Cayley's method of generating functions had been pushed to its furthest limits, and had become indeed a fruitful source of error, the symbolic treatment in the hands of Aronhold, Clebsch, Gordan, and others, was successful to an extraordinary degree, not only in showing how the errors arose, but also in establishing the facts. No one perhaps has imported more new ideas into the subject than Sylvester, and here also Hammond Franklin and others enabled progress when a block upon the line looked inevitable. When the theory began it was entirely algebraical, and limited in its scope to multinomial quantics. Its proper place in mathematical theory as a principle connected with the theory of groups was determined by Sophus Lie, Klein, Poincaré, and other workers on the continent of Europe. In particular the invariance of differential and functional expressions in respect of certain groups of substitutions finite or infinite, continuous or discontinuous, was studied and foundations solidly laid, principally by Sophus Lie. In Great Britain, what may be called the advanced work in the theory is almost entirely due to Forsyth, who has treated the differential invariants connected with algebraic curves and surfaces with success.

The work under review is entirely concerned with the Invariants Theory of Quantics, a theory which, it should be said, is of fundamental importance in regard to the more general theory. Most of the invariants theories that have been studied, including Sylvester's theory of Reciprocants, and those recently treated by Forsyth, depend in large measure upon the Invariant Theory of Quantics; the same partial differential equations and the same fundamental forms crop up time after time, and the same notions and methods are found effective in making progress. Original ideas and successful methods are available over the whole field.

Previous to the appearance of the present book, the only works of note on the same subject in the English language were Salmon's *Higher Algebra* and Elliott's *Invariant Theory of Quantics*. The former is a resumé of the work of Cayley, of Sylvester, and of the School of Trinity College, Dublin, with a slight sketch only of the symbolic method. The latter is a fairly exhaustive account of the progress that has been made by means of differential equations and operations and generating functions, to which its author has largely contributed; it does not touch on symbolic treatment.

We have here what the authors modestly describe as "an English introduction to the symbolical method." It does not pretend to be an exhaustive treatise—a series of volumes would be necessary for that—but an attempt to give sound instruction and information to students who wish to study the classical memoirs and themselves take part in research. In this it is quite successful, for it shews step by step how the authors have, within the last years, made discoveries of the greatest importance entirely by the symbolic method.

After an elementary exposition in the first five chapters we are given an account of Gordan's theorem, and the connected work of Jordan, Hilbert, and Stroh. The main object is to prove that for any system of quantics there exists an arithmetical system of invariants containing a finite number of forms, of which all other invariants are rational integral functions. In distinction to this there exists an algebraic system of forms finite in number, of which all other invariants are algebraic functions; this is a less difficult theory, on which the chief authority is Forsyth, who, in masterly memoirs in the *American Journal of Mathematics*, has practically made it complete by applying the theory of linear partial differential equations. Many mathematicians are of opinion that there is no theoretical advantage in obtaining an arithmetical system, and that its determination is of sentimental interest. Be that as it may, there is no doubt that the search after the arithmetical system has given birth to refined and beautiful mathematical work, which has had a large measure of success. The authors base the proof that the number of irreducible forms is finite upon a property of Diophantine Equations that was first brought into prominence by Hilbert; the theorem states that "the number of irreducible solutions in positive integers of a system of homogeneous linear equations is finite." The authors, however, fail to notice the most important fact in regard to such a system, viz., that it is invariably associated with a syzygetic theory; it is this circumstance that has enabled the construction in several instances of a syzygetic theory of ground forms appertaining to groups of transformations. It would have been of much advantage to students of this book if syzygetic theories for some simple Diophantine systems had been given in detail, for although their existence is implied by the Lemmas of Hilbert, the real point is likely to be missed in the absence of an explicit statement with illustrations. The manner in which the Lemmas in question are set forth in Chapter IX. otherwise leaves nothing to be desired.

The tenth and eleventh chapters give a satisfactory account of the geometrical interpretation of binary forms and of the theory of "apolarity." For references in regard to the latter theory the reader is pointed to Meyer's *Berichte*, an exceedingly valuable work that should be in the hands of every student of invariant theory. After an account of the symbolic method of treatment of ternary and quaternary forms we come to the most novel and important part of the treatise, the chapters on "Types" and "General Theorems on Quantics." The authors here are on the frontiers of knowledge, and make some epoch-making discoveries. By a consideration of the covariant types of a system of binary quantics, of which none of the orders exceeds n, they determine the maximum order of an irreducible covariant of such a system. If $n=2^\lambda+n_1$, the order in question is shewn to be

$$(\lambda-1)2^\lambda+n_1(\lambda+1)+2,$$

and the further observation is made that if the results for perpetuants are absolutely accurate (and of this there is practically no doubt) the maximum order is always reached, even for a single quantic of order n, except for the case $n=3$. This beautiful result is one of the most important ever made in the theory of binary forms, and would give distinction to the book if it contained nothing else of value. In Chapter XVI., moreover, is given an account of the "quantitative substitutional analysis," a powerful method recently invented by Mr. Young. It is applied to prove in a single page the Theorem of Peano, which asserts that "every type of the binary n-ic which does not furnish an irreducible covariant for a system of n n-ics is reducible, with one possible exception." Similar results are obtained for forms in any number of variables; thus, for instance, it is shewn that all covariants of any number of ternary n-ics may be obtained by means of Aronhold operators from the system for $\binom{n+2}{2}-1$ ternary n-ics, with one possible exception. The substitutional analysis of Mr. Young is an operative instrument based upon the theory of groups, which is sure to have an important future in connection with modern algebra; the authors have clearly explained it, and placed in the hands of students a weapon of research more useful, I believe, than any other that has been invented up to the present time for effective work in this part of algebra.

A book of less than 400 pages cannot possibly cover more than a small portion of the subject matter of invariant theory; it is therefore not surprising to find that the subject of the invariants of the orthogonal substitution is not touched upon. The methods of the authors are singularly well adapted to their consideration, which perhaps some day they will take up.

No better book can be placed in the hands of the rising generation of mathematicians.

Feb. 24, 1904. P. A. MacMahon.

Théorie Nouvelle des Fonctions, exclusivement fondée sur l'idée de nombre. Gustave Robin. **Oeuvres scientifiques réunies et publiées** par Louis Raffy. (Gauthier-Villars.) Paris, 1903.

The present volume of some 200 pages contains the work in pure mathematics of Gustave Robin, who has hitherto been known as the author of a number of memoirs on applied mathematics and physics which are to form two subsequent volumes of the present edition of his works.

We learn that the theory now presented was elaborated during the last years of Robin's life. The fundamental idea had occurred to both author and editor when they were students together: it dominated the course on the elements of analysis which Robin gave in 1892-1893 to students in physics at the Faculte des Sciences at Paris.

Sensitive and diffident, Robin never published his work. In fact, the editor says in his preface: "*Sachant qu'il ne consentirait jamais à l'écrire lui-même, je le décidai, pendant les vacances de* 1894, *a m'en exposer chaque jour, suivant ses reflexions du moment, telle ou telle partie: le tout fut pris en notes. C'est à l'aide de ces notes qu'a été établi le texte de la Théorie nouvelle des fonctions.*"

A volume so written must be necessarily fragmentary: lacunae must occur which we should wish to see filled up: we expect to find developments omitted which belong to the main theory. The care of the editor has, however, given us a theory which, so far as it is independent of the theory of functions of two independent variables and all the subsidiary integral investigations associated with the name of de la Vallee-Poussin, contains most of the results which we can legitimately demand.

But at the outset the reader will undoubtedly feel a sense of dismay. In the preface we read that MM. Robin and Raffy desired to build up a theory from which "the so-called (*prétendus*) irrational numbers, pure symbols of incommensurable geometrical magnitudes, should be excluded." In the introduction we are told that the authors "will put forward the reasons which make us reject from our science, some as useless, some as unjustifiable, the ideas and developments which occupy the central place in modern treatises on analysis." Finally, towards the end of the introduction: "By the admission of those very authors who make the theory of the so-called irrational numbers the basis of analysis, there are no *numbers* to which we can give such a name; and when, for the basis of this theory, we say that certain operations define an irrational number, we express nothing. We are only guilty of an abuse of language in that we forget the meaning of the word *define*." And once more: "The traditional confusion between numbers and magnitudes is the vice of modern mathematics." "Modern analysis is *un jeu stérile de l'esprit*." Thus the three great presentations of the theory of irrational number, associated respectively with the names of Weierstrass and Pincherle, Cantor and Heine, and Dedekind, are to be replaced by the ideas which MM. Robin and Raffy put forward. The authors certainly do not lack boldness: some of the statements which have been quoted make us turn to the *imprimatur* of the title-page, 'sous les auspices du ministère de l'instruction publique,' to be quite sure that we have not to deal with that scientific nuisance, 'the crank.'

But when we turn to the theory proper, there is no development of the controversial attitude of the introduction. The position taken up is stated clearly, the definitions are lucid, and successive propositions are built into the general theory with elaborate care.

According to the fundamental idea, $\sqrt{2}$ is the measure of a geometrical magnitude, the ratio of the diagonal to the side of a square. This is a concept not derived from the idea of pure number. If we restrict ourselves to the latter class of ideas, we must consider the decimal fraction 1·4142 ....

This fraction, if we stop at any decimal place, is a rational number. The sequence of such rational numbers is such that, after a time, any one means the same as any succeeding one to our finite senses. The difference, for instance, between the fraction taken to the 1,000,000th and that to the 1,000,001th decimal place cannot be measured by these senses. If then we have a real convergent sequence

$$a_1, a_2, \ldots, a_n, \ldots,$$

which does not tend to a rational limit, we may say that it defines a number a, where a means any term of the sequence after the nth, n being sufficiently large. The sequence is termed an *irrational sequence*, and the interval between two numbers a and b, so defined, is termed an *irrational interval*.

If to each number which belongs to any interval (a, b) we make correspond a known convergent sequence, we say that the aggregate of all these sequences *defines a function* of the variable x in the interval (a, b). Thus we make our function correspond to each rational number x, but not to each convergent sequence $x_1, \ldots, x_n, \ldots,$ between a and b. For if the latter sequence were irrational and $f(x)$ designated the number corresponding to it, we should mean by x no one of the terms of this sequence nor even any (rational) number. The numbers x between a and b for which $f(x)$ is defined do not therefore form a *continuum*.

Such are the fundamental differences between M. Robin's theory and those theories which have become classic. The first startling result to which the new theory gives rise is that *a function, finite within any interval, does not in general attain its upper and lower limits.* Thus $1-(x^2-2)^2$ never attains its upper limit unity.

The ideas of mean oscillation and integrable functions, and the condition for integrability, accord in the main with the classic theories.

The definition of uniform continuity is stated as follows:—$f(x)$ *is uniformly continuous in the interval* (x_0, X) if, given a positive fraction ϵ as small as we please, we can determine a number α such that for each pair of numbers x and x', which belong to the interval and differ by less than a,

$$|f(x')-f(x)|<\epsilon,$$

where by $f(x')$ and $f(x)$ we mean two terms (each of sufficiently high order) of the convergent sequences which define $f(x')$ and $f(x)$. On the other hand $f(x)$ is *continuous for the value* x if we can calculate a positive number a such that

$$|f(x+h)-f(x)|<\epsilon, \text{ when } |h|<a.$$

As a result *a function can be continuous for every number* x *belonging to an interval without resting finite in this interval.* (There appears to be an error in the explanation of the example given.)

In Chapter V. the idea of differentiation is introduced. Let $k=f(x+h)-f(x)$: then if the sequence $\frac{k}{h}, \frac{k'}{h'}, \frac{k''}{h''}, \ldots$, in which the denominators tend to zero, is convergent and equal to every analogous sequence, we call each term of the sequence of sufficiently high order the *derivate* of $f(x)$ and denote it by $f'(x)$.

From this definition arises the theorem that *if the derivate of a function is zero for all values of* x *within an interval, the function is* **not** *necessarily constant in this interval.* And it is possible to construct functions which are *continuous* in every interval, which cannot be decomposed into a definite number of sub-intervals where they are constant, and which have nevertheless a derivate zero throughout the whole interval.

We have to introduce the idea of uniform differentiability to obtain propositions analogous to those of the classic theories: $f(x)$ is *uniformly differentiable* in an interval (a, b) if for any fraction ϵ we can determine a number a such that

$$\left|\frac{f(x')-f(x)}{x'-x}-f'(x)\right|<\epsilon, \text{ so long as } |x'-x|<a,$$

x' and x lying within the interval.

When a function possesses a derivate but is **not** uniformly differentiable, a, instead of being constant, depends upon x. As an example,

$$f(x)=x^2 \sin \frac{1}{x} \; [f'(0)=0]$$

is not uniformly differentiable in an interval including $x=0$, although its derivate is everywhere finite.

A function $f(x)$ which is uniformly differentiable within an interval (a, b) is continuous in this interval, as is also its derivate. If the function vanishes for a and b, its derivate vanishes within the interval. The first theorem of mean value holds for such a function, and for uniformly differentiable functions the derived set of propositions accords with the usual theory.

But the first theorem of mean value holds for other functions than those which are uniformly differentiable, and in this connection we should like to see further investigations given.

In Chapter VII. the author discusses series of functions, their continuity and their derivates. He gives $\sum_n \frac{\sin(n!\,2\pi x)}{n!\,2\pi}$ as a continuous function which has a derivate for no value of x. Chapter IX. is concerned with power-series. In the succeeding chapter we have a somewhat fragmentary discussion of integrals and primitive functions followed by an investigation of singular integrals and Fourier's series. Here we expect and find some curious results. *Non-zero functions exist of which the integral vanishes for every value* x *comprised in the interval where the functions are defined.* And again, *every continuous function has an infinity of primitive functions, and only those primitive functions which are uniformly differentiable belong to the set* $\int_{x_0}^{x} f(x)\,dx + \text{const.}$

The final chapter of the volume gives scattered notions and theorems in the domain of functions of two variables. It is the natural extension of the previous ideas.

A somewhat long account of M. Robin's book has been given because of the undoubted interest of its novel point of view. We must regret Robin's early death and the unfinished state of his work: perhaps M. Raffy will do more than is comprised in the present volume to complete his friend's investigations. It is very doubtful whether the new theory will ever displace the traditional point of view: it deserves, however, to be read in that the very ss of its propositions may make the reader consider again the difficult ch underlie the theory of functions of a real variable and the theory of n which it is based. E. W. BARNES.

ntroductory Treatise on Lie's Theory of Finite Continuous rmation Groups. By J. E. CAMPBELL, M.A. Pp. xx, 416. 1903. on Press.)

book is properly, though modestly, called introductory, because it starts *io.* It is, however, comprehensive. Any pretence to be exhaustive it disclaims: but it goes a long way, and is very thorough, never shirking difficulty or sacrificing generality in the supposed interest of the beginner. It is, of course, on *Lie's* theory of continuous groups: there is no other theory of them. But its presentation of the theory is by no means borrowed from any previous work.

Sophus Lie was a genius. A thinker of marvellous power, vast energy, unrestrained egoism, he was inspired with a new grand idea, saw far as to whither it could lead, laboured enthusiastically, and was always at the forefront with it, never content that any other should be alongside except as almost part of himself. In these modern times, when so many skilful and highly trained mathematicians are ever keen to discover an opening for fruitful research, it testifies strongly to the pre-eminence of the man, that the highly developed theory which has resulted from his initiative should still owe relatively (though not absolutely) little to workers not directly associated with himself. Such workers are busy, however; and now, alas! there is no Lie left to forge ahead of them.

Great works exist on the elaborated theory. Their authors produced them with and for Lie himself. The time has come for treatises without the impress of direct guidance from the master, written by his independent followers. Of these Mr. Campbell is one. He has saturated himself with Lie's ideas, has thought them out again for himself, has done good work in investigations connected with them, and now gives us the benefit of seeing how the train of them looks most natural to him. It is not only to the English reader that he thus does a service. The book will be a standard work elsewhere, as the first considerable one on the subject regarded as classical and not as the property of its originator. But to the English reader his service is specially great. A translation of Lie-Engel or Lie-Scheffers would, though ponderous, have been acceptable as removing merely linguistic barriers in the way of our familiarization with the study. We wanted more, however, a book neither too short nor too expanded, one from a point of view accessible to those brought up as we have been, one with the kind of condensation which we appreciate, with the insular traits to which we are accustomed, and with shortcomings, if any, such as we know how to discount. And we now have a good one.

It is needless in this notice to expatiate on what transformation-group theory is. Earlier reviews in the *Gazette* have done so. Mr. Campbell begins with two excellent introductory chapters devoted to making the matter clear. They will be most valuable as furnishing ideas, even to those who have no intention of striving to master much of the elaborate analysis connected with later developments. They flow easily when the slightly too condensed first two pages are passed. Then follow three powerful chapters on the infinitesimal operators of a group, and the three fundamental laws as to their relationship and combination. Much here is unavoidably difficult. The treatment has considerable novelty. The proof of the converses of the second and third laws is briefer, and more readily convinces, than other proofs which have been elaborated. Much use is made of an "exponential" theorem (§ 50), which is an important one due to the author himself (later but independently also arrived at by Pascal). At an earlier point (§ 40) the all-important first law is proved with elegance, but unfortunately stated with some obscurity owing to excessive brevity. The reader must look on twenty-eight pages to § 59 for help to realize it, and yet fourteen pages more to § 67 to find an explicit statement of what it entails. This is almost a unique case in the book of lack of helpful explanation at the outset. The author's ordinary practice is the admirable one of preparing the student to realize the object of a difficult piece of analysis which is coming, by a few well-chosen words in advance descriptive of the aim and purport of the investigation.

After these chapters follow a good short one on complete systems of linear differential operators, an adequate discussion of the admission of known groups by differential equations, and a very interesting chapter on invariant theories. Primitivity, similarity, isomorphism, etc., of groups follow: then the theory of the construction of groups with known structure constants. Everything is at once presented with full generality: not led up to from the particular. The particular is only given by way of example. Pfaff's equation and Pfaffian systems next appear, and the author is led on to another subject, on which he speaks with special force and interest: that of contact transformations. This he at length leaves to describe, in chapters xxi. to xxiv., how all possible groups in not more than three variables can be obtained. Of these four chapters the first offers attractive reading, even to those to whom the three dimensional problem is repellent from its tediousness. A final chapter, with a note acknowledging indebtedness to Scheffers, but containing also some novelty, is devoted to certain linear groups connected with higher complex numbers.

The style throughout is terse and direct. Misprints are neither numerous nor absent. The printing is very clear: indeed, the Oxford Press is to be congratulated on a success in making suffixes legible, which is very unusual, and is of great value in a book on a subject like this, which requires them to be many, and generally triple. There is a good index and a very full and useful table of contents.

At times one would have liked the examples to be in a different type, or indented in the text. As a general rule, division into short articles has removed the danger of obscurity; but there are exceptions. The lengthy § 44, for instance, contains many things; and had it been clear at a glance where one thing ended and another began, the reader would have been helped.

E. B. ELLIOTT.

Elements of the Theory of Integers. By J. BOWDEN, Ph.D. Pp. x, 258. (Macmillan.)

The reader of this book is at once struck by the fact that the author writes "ar," "siv," and "thot" for "are," "sieve," and "thought" respectively, while retaining the usual spelling of "one" and "two." The same spirit of unsatisfactory compromise unfortunately pervades his analysis, and it is difficult to see what good purpose is served by the publication of a work of this kind. It is too abstract for the ordinary reader, and not nearly logical enough to satisfy a specialist. Professor Bowden appears to have consulted his authorities without being able to estimate their relative importance, and he has not taken up any consistent attitude of his own. He makes no reference to Dedekind's *Was sind und was sollen die Zahlen?*, and is presumably unaware of its existence; otherwise he would hardly have made $b+a=a+b$ an axiom. The fact that he does this, while spending two pages to prove that the sum of two even integers or of two odd integers is even, while the sum of two integers, one of which is even and the other odd, is odd, illustrates his vague attitude and lack of the sense of proportion. It may be added that counting is discussed after propounding this so-called axiom, the only justification given for which is derived from collecting two groups of objects into one. To add to the confusion, we are told in a foot-note that "the idea of number is prerequisit [*sic*] to, not derived from, the idea of counting": and this after saying in Art. 1 that "to every group of objects belongs a number, the number of objects in the group"; a statement followed by numerous propositions, with proofs based upon the unanalysed, and certainly not primitive conception of "a number [*Anzahl*] of objects." Contrast with this the conscientious minuteness of the following:

"In symbols $a=a$.

"A longer statement of the same principle is: *The number of objects in a group is the same as the number of objects in that group.*"

It is only fair to say that some parts of the book, such as the chapter on zero and negative integers, are comparatively satisfactory; but it contains nothing which has not been done, and done better, before.

G. B. MATHEWS.

Geometrie der Dynamen. Die Zusammensetzung von Kräften, und verwandte Gegenstände der Geometrie. Von E. Study. (Leipzig, Teubner, 1903.) Pp. 603. M. 21.

The first half of this book was published in 1901, and consists of two parts treating of the composition of forces and wrenches, first by geometrical and secondly by analytical methods. The second half and third part was published last year, and contains a new sort of line geometry which the author has invented, with numerous and far-reaching applications.

These pages contain a kernel of sterling value concealed within an immense husk of almost impenetrable material which would prove trying to even the strongest intellectual teeth; but the kernel, when once reached, is one of those pieces of mathematics which give a thrill of satisfaction to all who appreciate them, by their beauty and simplicity in addition to their enormous power.

This principle, which was explicitly stated by Mr. Homersham Cox twenty-two years ago, occurs at the end of Prof. Study's second part. Many of the theorems presented geometrically in the first part are deductions from it, and are consequently almost unintelligible where they occur. The author takes the greatest care to prepare for every possible exception or particular case of a metrical nature, and this leads him to invent a bewildering number of new names, many of which connote overlapping classes of things. Thus, for example, two lines not in the same plane form an "impulsor," and two lines which are not perpendicular nor at infinity form a "motor," and so on. It is doubtful whether the passion for exact definition which leads to such a picking to pieces of the subject tends to any true advance: it certainly checks the progress of the reader. The third part of the book is concerned with a new set of six line-coordinates designed to overcome difficulties arising at infinity: of this part, containing doubtless much important matter, no notice is taken here, but the reader is referred to an enthusiastic review in the January number of the *American Bulletin*.

The mathematical principle which led to the writing of this book is best presented in the first instance as a simple device whereby much of the labour of calculating with line coordinates may be avoided, and known results may be extended from two dimensions to three. A line through a fixed point (the origin) is determined in reference to a given frame by three direction cosines, l, m, n, satisfying the single relation

$$l^2+m^2+n^2=1,$$

and any line in space is determined by six coordinates l, m, n, l', m', n' satisfying the preceding relation and one other, namely

$$ll'+mm'+nn'=0.$$

It follows from this that if l, m, n are regarded as functions of a parameter, their differential coefficients *may* be regarded as the second set of three coordinates of a line in space, and further, of *any* line having the direction l, m, n. Anyone can follow out this idea for himself; it leads to the rule: by differentiating the solution of a problem concerning lines through the origin the solution is obtained of the corresponding problem for lines in space. For example, the last three coordinates of the common normal of two lines are deduced by simply differentiating its direction cosines. Again, the well known theorem that two polar reciprocal triangles are in perspective leads to the beautiful theorem, discovered independently by Prof. Morley of Baltimore and Dr. Joh. Petersen of Copenhagen in 1898, that the three common normals of opposite sides of a skew rectangular hexagon have a common normal. Conversely, Study's "Übertragungsprincip" and Petersen's "Principe de duplication" can be based in a purely geometrical way upon this theorem proved independently, or on the equivalent theorem, employed by the latter writer, that simultaneous increments of the sides and angles of a spherical triangle are proportional to the sides of a certain skew right-angled hexagon. For more detailed explanations and exact references the reader is referred to two papers on this subject in the *Messenger of Mathematics* of 1902.

But this principle is not merely an ingenious device. It lies at the foundation of all line geometry, *including that of* "elliptic" and "hyperbolic" space. The three types of *absolute* can be written in one form,

$$\epsilon^2(x^2+y^2+z^2)\mp t^2=0,$$

where $\epsilon^2 = -1$ for hyperbolic, $\epsilon^2=0$ for Euclidean, and $\epsilon^2=1$ for elliptic geometry. The "rank equation," upon which the metrical relations of lines depend, is

$$l^2+m^2+n^2+\epsilon^2(l'^2+m'^2+n'^2)=0.$$

This can be written in the form

$$(l+\epsilon l')^2+(m+\epsilon m')^2+(n+\epsilon n')^2=0,$$

resembling the absolute of spherical geometry. In this lies the ultimate reason for the existence of the principle of extension.

In these equations ϵ is a *number*, real or imaginary. The next step in logical sequence takes its origin in one of Clifford's suggestions, to whom so many of the ideas in this region can be traced back. He introduced a *symbol* ω, obeying the ordinary laws of algebra, and in addition the equation $\omega^2=1$. With this he coupled two vectors together to form a "bivector" and two quaternions to form a "biquaternion." Using the same symbol, the *dual coordinates* of a line (l, m, n, l', m', n') are defined in the first instance to be $l+\omega\epsilon l'$, $m+\omega\epsilon m'$, $n+\omega\epsilon n'$, and it is then evident that $\omega\epsilon$ may be replaced by a single symbol obeying the same laws as ϵ in the three cases. In hyperbolic and Euclidean geometry ω may be omitted without confusion; in elliptic geometry ϵ is omitted, being unity, and then the presence of ω distinguishes the "dual coordinates" of a line from the "Clifford parallels." This principle was first clearly formulated in 1882 by Mr. Homersham Cox in his paper "On the Application of Quaternions and Grassmann's Ausdehnungslehre to different kinds of Uniform Space" in the *Cambridge Philosophical Transactions*, vol. xiii, p. 69. On p. 112 he says, "We see then that in all three kinds of geometry the properties of lines in space can be derived from those of lines meeting in a point if we write . . . $l+\omega l'$, etc., for its direction cosines . . . where in the first kind of geometry $\omega^2=-1$, in the second $\omega^2=0$, and in the third $\omega^2=1$."

With this quotation in mind, it is interesting to read, on the last page of the book, how inadequately certain English mathematicians have attempted to expound Clifford's ideas. They are regarded as "lamentable sacrifices to a system of education which is still in vogue in England"! The germ of truth in the last phrase serves only to accentuate its inappropriateness in connection with the work of Cayley, Hamilton, Clifford, and others. The subject of Universal Algebra, with all its geometrical interpretations, received much of its development in this country, and while assistance is welcomed from all quarters, it should not be dealt out with a too heavy hand. R. W. H. T. HUDSON.

UNIVERSITY OF LIVERPOOL.
February, 1904.

Technical Mechanics. By E. R. MAURER, Professor of Mechanics in the University of Wisconsin. Pp. vi, 382. Price 17s. net. 1903. (New York: J. Wiley & Sons. London: Chapman & Hall.)

Professor Maurer's work may be regarded as intermediate between an elementary text book on the one hand, and the standard treatises on special branches of mechanics on the other. Presupposing in the reader a certain amount of elementary knowledge, the author discusses statics, some simple cases of attractions, and the plane motion of a particle and of a rigid body.

In statics, graphical and analytical methods of proceeding are developed side by side. The graphical treatment of the conditions of equilibrium of forces in one plane, and of forces in space, may be alluded to as specially noteworthy.

Professor Maurer has indicated the correct aim of a book on Technical Mechanics in his preface in the following sentence: "On the theoretical side practically all the subjects treated have a direct bearing on engineering problems. On the applied side no attempt is made to present fully any one subject, such as . . . the balancing of rotating systems, for the object in view was the illustration of the use of the principles of mechanics, and not treatments of . . . balancing. . . ."

In carrying out this design much skill and judgment is shown in the selection of illustrative examples of practical interest, without allowing the question of principle under discussion to be obscured by mere details. Minute practical details in a text-book are apt to be the details of the practice of yesterday, not to say of the day before.

Unfortunately, the treatment of particular topics is not always maintained at the same level of excellence as the planning of the broad outlines. The following extract (page 190) illustrates my contention that a student, and not necessarily, I think, a merely careless or hasty student, might easily obtain erroneous notions from several paragraphs of the work.

In investigating expressions for the tangential and normal accelerations of a particle moving on a plane curve, the author takes the axes of x and y parallel to the tangent and normal at P, and continues "*with axes thus chosen* (author's italics) it is plain that the tangential and x components and the normal and y components are equal, or

$$a_t = a_x = \frac{dv_x}{dt}, \quad a_n = a_y \text{ (sic) } = \frac{d^2y}{dt^2}.$$

Now

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\,\frac{dx^2}{dt^2} = \frac{d^2y}{dx^2}\,v_x^2;$$

also $\frac{d^2y}{dx^2}$ equals the curvature of the path at P. Denoting the radius of curvature at P by ρ, and since $v_x = v_t = v$,

$$a_t = \frac{dv}{dt} = \frac{d^2s}{dt^2} \text{ and } a_n = \frac{v^2}{\rho}."$$

The statements in the penultimate sentence quoted happen to be true because $\frac{dy}{dx} = 0$ at P, but the coincidence cannot be considered as equivalent to a proof. In fact, the reasoning would equally serve to prove $a_n = v^2 \frac{d^2y}{dx^2} \times f$, where f is any function of $\frac{dy}{dx}$, which becomes unity when $\frac{dy}{dx} = 0$.

Notwithstanding a certain absence of precision in expression, particularly as regards kinetics, of which the foregoing extract gives an unfavourable instance, the excellently chosen examples, the good and clear printing, and the conciseness of the treatment make the book decidedly an attractive one.

Professor Maurer, it is interesting to observe, adopts the "engineers'" units and proposes the terms "geepound" and "geekilogram" for the "engineers'" unit of mass.

On page 254, Ex. 5, $a^2/3$ should be $a^2/12$, and on page 260 $m\bar{r}\omega$ should be $m\bar{r}\omega^2$.

C. S. JACKSON.

An Elementary Treatise on Cubic and Quartic Curves. By A. B. BASSET. Pp. xvi + 255. 10s. 6d. 1901. (Deighton Bell.)

The author tells us in the preface that "the present work originated in certain notes, made about twenty-five years ago," and the general effect is one of discontinuity. The earlier chapters of the book mainly reproduce some of the results contained in the first half of Salmon's *Higher Plane Curves*; the methods of obtaining these results are novel. The treatment of Plucker's equations is specially striking, and equally unsatisfactory; the author would, in fact, have done better to have omitted his first four chapters and given a reference to Salmon's work in the preface.

Mr. Basset confesses to a liking for the use of anglicised Greek words, and remarks that mathematicians would do well to follow the example of the medical profession, and, when they require a new word, have recourse to the Greek language. Acting on this principle, he introduces into his text words such as autotomic, anautotomic, endo-dromic, exodromic, perigraphic, etc., etc. The second half of the book is a great improvement on the first: curves of the third order and of the third class respectively are duly classified (without, however, any reference to recent work in this direction) and some space is devoted to the special treatment of historical curves like the trisectrix, the cissoid of Diocles, Descartes' folium, and the witch of Agnesi.

Since the book is of a professedly elementary character, the classification of quartic curves, by means of the lines on a cubic surface and the use of Schlafli's notation, is not to be expected; the classification given is therefore naturally somewhat loose. In the later chapters the projection of tricircular quartics is discussed.

The whole question of plane cubics and quartics is nowhere seriously undertaken; elliptic functions are not used, and some of the analysis is of a very clumsy character The subject matter is so diffuse that a good index is indispensable to anybody using the book as a text book but the book cannot be recommended as a text book, so perhaps the absence of an index is immaterial The appearance of this treatise once again emphasises the necessity for a new edition of Salmon's *Higher Plane Curves*, brought up to date with the researches of the past twenty years. Now that the learned author is no more, perhaps some enterprising geometer will undertake the task practically, the only modern work on the subject is Fiedler's German translation of Salmon, and a comparison of the contents of the original work and its translation is sufficient to show how far the subject has grown in the last few years P. WORSLEY WOOD

Lectures on the Logic of Arithmetic. By M E BOOLE (Clarendon Press) While disagreeing with some of the doctrines laid down in this book (and humbly acknowledging that we may be quite wrong in doing so), we can yet heartily recommend it to all those who are anxious to gather in from whatever source any ideas which may add to the efficiency of their teaching Many of us who are "too far in years to be a pupil now' to the philosophy of the authoress, will rejoice to see that she reaches the same results by highly philosophical reasoning that we have found out by the experience of the class room We advise all to read on past whatever they are inclined to disagree with, and come to the practical bearing of the precepts laid down The "mental picture' method is of wide application The illustration of the nature of curves by the study of the dog following the man, the nature of the minus sign by the nature of "change" in shops are cases in point If any teacher of higher forms should be irritated by the childish way in which the dialogue is put, he may easily *mutatis mutandis* adopt the method used

A New Geometry for Junior Forms. By S BARNARD, M A , and J M CHILD, B A (Macmillan & Co 2s 6d) Among the additional sections which have been incorporated with the easier parts of the New Geometry to form the present volume, are a set of Introductory Exercises on Experimental work, an explanation of the Construction of Scales and a well illustrated account of the Forms of *cube, cuboid, tetrahedron, pyramid, prism cylinder and cone* These, and especially the last, add much to the usefulness of the book It was a wise step of the Universities to include the forms of the simpler solids in their elementary syllabus. Under the old method, a pupil who left without reading XI 1 21, stood a chance of never receiving any systematic instruction in the simpler solid forms From our own experience, we believe few things so well calculated to arouse attention and sustain interest than the inspection and construction of models. The authors give the *nets* as well as the perspective views, and suggest the construction of others of varied dimensions for solids whose form is given in perspective and dimensions in words Such exercises are beneficial, and from these the pupil may easily advance to easy orthogonal projection for representing *in plano* the solid he has planned and constructed

Elementary Geometry. By CECIL HAWKINS, M A (Blackie & Son) A useful treatise on the subject matter of Euclid I IV , in which the author seems to be inspired by a wish to go as far as possible in the direction of the reformers, and has no wish to make his treatise look like Euclid brought up to date. We are introduced to symmetry (axial and central) and images There is an extensive and varied collection of exercises both in Theory and Practice

Elementary Geometry. Section III. By F. R. BARRELL, M A., B.Sc 1s. 6d. (Longmans, Green & Co)

This section deals with the subject matter of Euclid's Book XI , and gives, in addition, a careful treatment of the Cylinder, Cone, and Sphere It can be heartily recommended. We have drawn attention, in receiving the previous sections, to the great care and skill shown in the diagrams. These qualities are still more marked in the third section, as Solid Geometry gives more scope for their exercise Plato's five regular solids get very scant attention, being merely noticed in some exercises, but the space gained by omitting them is perhaps as well, or more usefully occupied by the Sphere, which deserves to have its properties made part of every school course of Solid Geometry. Areas and Volumes are satisfactorily treated.

An Introduction to the Study of Geometry. By A. J. PRESSLAND, M.A. 1s. (Rivingtons.)

In this little book we find the same excellence in execution of diagrams and arrangement of matter as in the author's *Geometrical Drawing*, a work which advocated experimental treatment when it was not so fashionable as it is now. The theorems selected for experimental treatment are calculated to arouse the interest of the young student, and to lead him to try further mathematical experiments on his own account. There is a useful introduction to Graphical Arithmetic.

Dr E. Bardey's Anleitung zur Auflösung engekleideter algebraischer Aufgaben. By F. PIETZKER. (Teubner.)

This seems a new edition of an old work of which little remains except the title. It aims at showing how problems wrapped up in ordinary language are to be translated into that of Algebra. The examples are taken from widely different sources, Trigonometry, Astronomy, Mechanics, and Physical Science.

Scale and Protractor for Blackboard Drawing. Designed by A. J. PRESSLAND, M.A. Post free, 2s. 6d. (Made and sold by W. Brunton, West Silverwells, Edinburgh.)

This serviceable instrument has three scales marked on it. On one face degrees at intervals of 10, and a line scale extending on each side of the zero to 14 inches at intervals of one inch. On the other face it is graduated to 35 centimetres on each side of the zero at intervals of 5 centimetres. We have tested the accuracy and found the graduations quite fine enough for board work, at the same time that the stoutness of its make enables it to stand the wear and tear of a school instrument. We find it very useful. E. M. LANGLEY.

A Course of Pure Geometry. By E. H. ASKWITH. Pp. xii, 208. 1903. (Cambridge Univ. Press.)

This book consists of a preparatory course of lessons for such students as wish to specialise in Pure Geometry, as well as for those who merely require some general inkling of the modern methods in that subject. The author acknowledges his indebtedness to Messrs Casey, Lachlan, and J. W. Russell, as, indeed, must any writer on Pure Geometry for some time to come. But the course of lessons shews no slavish adherence to the methods of any other author, and in a measure may be said to mark an advance from the pedagogic point of view. For Dr Askwith is careful to point out that the distinction between pure and analytical geometry "is one of method rather than of idea," and an occasional digression to view the matter of a section from the analytic standpoint is no doubt fruitful in result. There can be no real divorce between the two when we admit the doctrine of continuity. We have no doubt that this text book, with its excellent collection of questions from Tripos and other papers, will be found to "fill a gap."

Annuaire pour l'An 1904. Published by the Bureau des Longitudes. Pp. 732, 116. 1 fr. 50 c. 1903. (Gauthier-Villars.)

Last year we drew attention to the new arrangements under which the *Annuaire* is published. To avoid "vain repetition," the usual tables are printed in such order that to have the complete set the reader must possess the *Annuaire* for two consecutive years. For instance, the present number contains in detail all the tables relative to Chemistry and Physics, and does not contain those referring to geographical and statistical data. The chemical and physical tables will not be given in the number for 1905, and will be replaced by those dealing with Metrology, Money, Statistics, and so on. So again, this year's *Annuaire* is *par excellence* the number for ancient calendars, and for a complete table of the elements of the minor planets. Next year it will be the turn of stellar parallaxes, double stars, stellar spectroscopy, etc. The essays, which have long been a characteristic feature of this publication, deal this year with the International Geodetic Conference, Aug. 1903, by M. Bouquet de La Grye, and an elementary explanation of the Theory of the Tides, by M. P. Hatt. We again call attention to the 848 pages of tables and letterpress for one franc fifty cents.

Arithmetical Types and Examples. By W. G. Borchardt. Pp. xii, 367. 1903. (Rivingtons.)

This is one of the first Arithmetics to endeavour to follow up the recommendations of the Committee of the Mathematical Association, and is an eminently successful exposition of the subject on the improved lines. The greater number of the examples are original, and those in the Test Papers are from papers set in public examinations. The exercises are each preceded by a model solution, with notes where needed; and another edition is to be published for the benefit of classes under teachers who object—as perhaps most of us do—to worked-out examples.

School Algebra. By J. M. Colaw and J. K. Ellwood. Pp. 432. 1903. (Johnson, Richmond, Va.)

This is an excellent introduction to the elements of Algebrr and as a special feature it develops from the outset the arithmetical basis of the subject. "Formulae and results are frequently tested by arithmetical applications, and in passing to new ideas numerical illustrations are often used." The Binomial and Remainder Theorems are early introduced—the former in the chapter on multiplication, the formal proof being of course deferred. Graphs are freely used throughout, and the questions set at the end of the sections are as far as possible drawn from the experiences of everyday life. A thoroughly careful little introduction to Algebra.

A Manual of Practical Mathematics. By F. Castle. Pp. vii, 541. 1903. (Macmillan.)

This is the necessary sequel to Mr Castle's previous volumes on Practical Mathematics, and brings the elements of the "higher" Mathematics within the grasp of the ordinary student. The treatment throughout is clear, and in every way justifies the author's hopes that the average student will be enabled to apply to practical purposes in a short time the principles of those portions of advanced mathematics of which he otherwise would long remain in ignorance. Among the sections in the book, for which Mr Castle, rightly or wrongly, claims the attribute of novelty, are the following. The graphic solution of quadratic equations—the identification of a plotted curve by celluloid strips on which the standard curves are displayed—solution of equations of type $T=a+by^n$—graphical treatment of $y=a\cos(\omega t+e)$, $y=a\sin(bx+c)$—Amsler planimeter, Fourier's theorem—obtaining slope of curve by set square and pencil—geometrical proof that

$$\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}$$

—the use of progressions to illustrate the Integral Calculus. We are inclined to think that this book will meet with a large sale.

The Schoolmasters' Year Book and Directory. 1904. Pp. lx, 1030. 5/- net. (Swan Sonnenschein.)

The second issue of the Year Book is larger than its predecessor, mainly owing to an additional fifteen hundred names in the Directory. The editor complains that incorrect information is not uncommon. The date of the foundation of certain schools were given as 1945, 1904, 184. and so on. There is a full list of the Education Committee of County and County Borough Councils. The publishers are to be congratulated on the success which has attended the appearance of this "excellent and indispensable" book of reference, and the editor on the absence of errors—as far as we have been able to test it.

Vorlesungen uber Algebra By Dr G. Bauer. Pp vi, 373. 1903. (Teubner.)

This volume was printed at the instance of the *Mathematischen Verein* of Munich as a compliment to Professor Bauer on the occasion of his 80th birthday, in November, 1900 From the portrait of the author, which forms the frontispiece, we gather that in spite of his advanced years his intellectual faculties are not yet dimmed The lectures are those given to students in their first and second year, and cover a wide range of subjects The first section contains the elementary properties of polynomials and rational fractions, the theory of symmetrical functions, elimination and the transformation of equations. On p 13 we have a construction for [illegible] by means of similar triangles, where n is needlessly restricted to positive integers The fundamental theorem that every algebraical equation has a root is proved in much the same way as in Chrystal, and is exposed to the same objection If for every value of $f(z)$ an h can be found such that mod $f(z+h)$ is less than mod $f(z)$, then $f(z)$ does not necessarily vanish For example Moritz gives $f(z)=\left(\frac{1+z}{2}\right)$, where z has no zero value, and yet for every value of z an h may be found such that $f(z+h)$ is less than $f(z)$

The second part deals with the algebraical solution of equations, the elements of the Galois theory, binomial equations Abelian equations, and a few properties of numbers Part III treats of the numerical solution of equations, the system of Graeffe as improved by Encke being that preferred by the author The principle is simple enough From the given equation a second is derived, whose roots are very high powers of the roots of the original equation The powers of the smaller roots may then be neglected in comparison with the powers of the larger roots The volume concludes with the theory of determinants and their applications, and two notes on continued fractions, and Lagrange's formula for obtaining the sum of the nth powers of the roots of a quadratic The book is clear and is easy to read Many beginners will find it exactly suited to their needs

I Tre Problemi Classici degli Antichi: studio storico critico. Problema Secondo La Duplicatura del Cubo. By Prof Bettino Carrara S J. Pp 114 1903 (Fusi, Pavia)

Professor Carrara has carried his researches into the three famous problems of antiquity one stage further by the publication of the section dealing with the Delian problem—the duplication of the cube The first ninety pages do not contain much that is not easily accessible to English readers in such books as Rouse Ball's *Recreations* and *Short History*, Heath's *Apollonius of Perga*, Allman's *Greek Geometry*, etc But in the sections dealing with the problem as treated by the mathematicians of the seventeenth to the nineteenth centuries we find much that is novel and interesting Here curiously enough Professor Carrara seems to be unaware of the part played by the English philosopher of Malmesbury in the course of his deadly feud with the mathematician Wallis For years Hobbes and Wallis had been disputing in terms as unseemly as they were biting, about the squaring of the circle, until Wallis gave Hobbes up in despair The latter tried in vain to induce Wallis to renew the argument He then adopted wily tactics He published in Paris under an assumed name what he fondly believed to be a solution of the famous Delian mystery It was in French, and his simple artifice succeeded, for Wallis came out of his lair and demolished the so called proof. Hobbes immediately claimed the authorship, defended it, making more blunders than ever in the process, and republished it in his *Dialogus Physicus, sive de Natura Aeris*—a vehement attack on Boyle and other friends of Wallis These he hated, for in the new Royal Society which they had just been instrumental in forming they had maliciously left out the redoubtable philosopher Hobbes once more was crushed by Wallis in the scathing *Hobbius Heautontimorumenos*, and was silent for some years But when he was over eighty years of age he published his *Quadratura Circuli, Cubatio Sphaerae, Duplicatio Cubi* Once more the tireless Wallis refuted the pretensions of Hobbes, but the latter, nothing daunted, republished his book, with answers to all objections, and the volume was dedicated, Professor Carrara will be interested to hear, to the Grand Duke of Tuscany. The battle went on for some years, and Hobbes fired his last shot at Wallis at the age of ninety! Those of our readers who ever come across the effusions of Wallis in this squabble will find that they are extremely rare. Professor Carrara's eighth chapter deals with the work of Mascheroni, Vargiù, Bonafolce, and Boccali in connection with approximations to $\sqrt[3]{2}$, etc. A few remarks on the use of the integraph bring an interesting brochure to a close. In one point the author is "sadly to seek." His quotations from French and German, even when only the titles of books, are often marred by inaccuracies. Thus we have "algebreischen" (p. 9); three mistakes in a quotation on p. 12; four in the title of Chasles' Aperçu (p. 18); commas have replaced dashes in the solution (p. 19); Allman, Greek Geometrie (p. 25); the same book figures as Greeck Geometrie (p. 32); Tunnery's Histoire general (p. 33); Archimcdis (p. 37); *perga for Perga* (p. 39); *contemporaim* (p. 42); goschichte (p. 43); wisseuschaften (p. 48); and so on. Even Apollonius is treated to an extra "p." It is hardly an exaggeration to say that thirty per cent. of the French accents are wrongly placed. This careless correcting of proofs mars what is otherwise a sound piece of work.

NOTICE.

THIRD INTERNATIONAL CONGRESS OF MATHEMATICIANS AT HEIDELBERG, AUGUST, 1904.

MEMBERS of the Mathematical Association are invited to be present at the Third International Congress of Mathematicians, which will take place at Heidelberg from Monday, August 8th, to Saturday, August 13th, 1904. The proceedings will commence with a reception at 8 P.M. on August 8th; there will be three general meetings, and also meetings of six sections, viz., (1) Arithmetic and Algebra, (2) Analysis, (3) Geometry, (4) Applied Mathematics, (5) History of Mathematics, (6) Pedagogy.

The centenary of the birth of C. G. J. Jacobi will be celebrated during the Congress.

There will be an exhibition of Mathematical Models (historical and modern), and an exhibition of the more important mathematical publications during the past ten years, to which contributions on loan are invited. Intending contributors to the exhibition of literature are requested to correspond with Dr. A. Gutzmer, Professor an der Universität Jena. Social meetings, a banquet, and an illumination of the Castle will also take place.

The price of a ticket for the Congress is 20 marks—additional tickets may be obtained by members of the Congress for their own party at 10 marks each person, which are available for the general meetings and festivities.

Further information concerning the Congress may be had from Professor A. Krazer, Karlsruhe i. B. Westendstrasse, 57.

ERRATA.

No. 43, p. 383: lines 13, 14. For $P\hat{T}R$ read $E\hat{T}R$.

„ $P\hat{Q}R$ read $A\hat{Q}R$.

Dele. and $R\hat{F}A = A\hat{Q}R$.

Front Cover, Jan. 1904. For Vol. III. read Vol. II.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) **For Sale.**

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I.-IX. Edited and Published by E. HENDRICKS, M.A., Des Moines, Iowa, U. S. A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie, Pa.)]

The Mathematical Monthly. Vols I. III 1859-1861 (interrupted by the Civil War, and not resumed) Edited by J. D RUNKLE, A M

Proceedings of the London Mathematical Society. First series, complete. Vols. 1 35. Bound in 27 vols. Half calf. £25

Cayley's Mathematical Works Complete, equal to new, £10. Apply, Professor of Mathematics, University College, Bangor

(2) **Wanted.**

Vols. I.-IV. *Mathesis* 1881 1884

The Messenger of Mathematics. Vols. 2, 15-20, 24, 25.

Tortolini's Annali. Vol I (1850), or any one of the first eight parts of the Volume.

Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

(3) Dr. Muir, The Education Office, Cape Town, will give Vol. 109, *Crelle's Journal*, to any member of the Mathematical Association whose set is without it.

BOOKS, ETC., RECEIVED.

Geometric Construction of the Regular Decagon and Pentagon Inscribed in a Circle. By H. H. LUDLOW. Pp 12. 1904. (Open Court Co., Chicago.)

Experimental and Theoretical Course of Geometry. By A. T. WARREN. Pp. viii, 264. 2s. 2nd edition. 1904. (Clarendon Press.)

Graphs and Imaginaries. By J. G. HAMILTON and F. KETTLE. Pp. 41. 1s. 6d. 1904. (Arnold.)

Lehrbuch der Analytischen Geometrie. Erster Teil By. O FORT. 7th edition. pp. xvii, 268. 1904. 4 m (Teubner)

Anschauliche Grundlagen der Mathematischen Erdkunde. By K. GEISSLER. Pp. vi, 199. 1904 3 m. (Teubner.)

Handbuch der Theorie der Cylinderfunktionen. By N. NIELSEN. Pp. xii, 408. 1904. 14 m. (Teubner.)

A School Geometry. Parts I.-IV. H S. HALL and F. H. STEVENS. pp. xii, 340, x. 4/6. 1903 (Macmillan). (Substance of Euclid Books I.-IV. and VI with Additional Theorems and Examples.)

A School Geometry. Parts I.-V. (Containing the substance of Euclid, Books I -IV., VI., with Additional Theorems and Examples.) By H. S. HALL and F. H. STEVENS. pp. xii, 340, ix. 4/6. 1904 (Macmillan).

Elementary Geometry Section III By F R Barrell Containing the subject matter of Euclid Book XI together with a full treatment of Volume and Surface of the Cylinder, Cone, Sphere, etc pp 285 360 1/6. 1904 (Longmans)

On the Transfinite Cardinal Numbers of Well ordered Aggregates By P E B Jourdain pp 61 75 (Phil Mag, Jan 1904)

Introduction to Quaternions By the late Professors P Kelland and P G Tait 3rd Edition Prepared by C G Knott pp xviii, 208 7/6 1904 (Macmillan)

First Lessons in Observational Geometry By Mrs W N Shaw pp ix, 148 2/ 1904 (Longmans)

Geometry on Modern Lines By L S Boulton pp viii, 126 2/ 1904 (Methuen)

Elementary Geometry By W M Baker and A A Bourne pp xxix, 477 Third Edition 4/6 1904 (Bell)

A Key to Elementary Geometry By W M Baker and A A Bourne pp 177 6/ 1904 (Bell)

Analytische Geometrie der Kegelschnitte By G Salmon Edited by W Fiedler Part II 6th Edition pp xxiv, 443 854 1903 (Teubner)

Sitzungsberichte der Berliner Mathematischen Gesellschaft pp 68 1903 (Teubner)

Der Geometrische Vorkursus in Schulgemässer Darstellung Edited by E Wienecke pp 97 1904 (Teubner)

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THE

MATHEMATICAL GAZETTE.

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OBITUARY.

THE REVEREND GEORGE RICHARDSON, who died at his son's residence in London on January 16th in his 64th year, will be missed by a large circle of friends. Born at Carlisle in 1840, he was educated at Chester, and on leaving school he obtained a Scholarship at St. John's College, Cambridge, at the age of 17. He graduated as Third Wrangler in 1860, the year in which Lord Justice Stirling was Senior Wrangler. In due course he was elected a Fellow of his College, and in 1867 he was appointed an Assistant Master at Winchester College by Dr. Ridding. From 1873 to 1899 he was Second Master at Winchester with charge of the scholars of the College.

Whilst at Cambridge he is said to have been able to read 8 or 10 hours a day without feeling any ill effects: and he carried the same amount of energy into his work at Winchester. His pupils felt that he had the art of putting things clearly, and a keen insight into the sources of their difficulties; and, echoing Goethe's words, they might have said: "Mit dem Genie haben wir am liebsten zu thun." Thoroughly successful in all that he undertook, he had a firm grip on the practical side of life; and, as sometimes happens, he was perhaps a little out of sympathy with the unsuccessful idealist. He told the present writer that he had "once spent several months in the study of metaphysics, and thought that there was very little in the subject." Though somewhat reserved, he was essentially "gemütlich," *i.e.* he had the comfortable and kindly good nature which made his pupils feel at their ease. It was, in fact, just this genial power and apparent absence of effort, which seemed so inspiring. The high places afterwards attained by many of his pupils in the Mathematical Tripos testify to the excellence of his teaching.

He felt the difficulties inseparable from the present public school system; and, with wide outlook on human nature, he thought there must be some way of surmounting them. On being presented with his portrait at Winchester in 1899, he strongly advocated the importance of educating boys and girls together. Perhaps the next generation will see the wisdom of the efforts already being made in the direction of co-education, and will endeavour to extend it.

Mr. Richardson was an original member of the Mathematical Association, founded as the Association for the Improvement of Geometrical Teaching in January, 1871, and in 1883 he was elected a Vice-President. At the annual meetings he took a leading part in the discussions, which are full of useful hints on the art of teaching Mathematics (see the Annual Reports, 1-19, some of which are still in print). In 1873 he wrote a book on Geometrical Conics. In 1887 he read a paper on "The Teaching of Modern Geometry," which was followed in 1894 by the publication of his "Modern Plane Geometry," written in conjunction with Mr. Ramsey on the lines of the Association's "Syllabus of Modern Plane Geometry." In *Gazette*, No. 32, there appeared a valuable paper on "The Trigonometry of the Tetrahedron."

Amongst his numerous friends was the late Mr. Wolstenholme, with whom he corresponded from time to time on mathematical subjects. He was also a life-long friend of Sir Robert Ball from the time when they were at School at Chester together. With his friend the Rev. W. D. Bushell (late Fellow and Tutor of St. John's Coll. and afterwards Mathematical Master at Harrow), he took a keen interest in the Cambridge Rifle Corps; it is not perhaps generally known that Mr. Richardson was once second for the Queen's Prize.

Mr. Richardson had great powers of organisation; and if he had been a classic instead of a mathematician, he would probably have been appointed Headmaster at one of the great public schools. His genial and exceptional powers adorned his profession, whilst at the same time he was

> "Rich in saving common-sense,
> And, as the greatest only are,
> In his simplicity sublime."

H. D. Ellis.

The following extracts from a letter to the Editor from Sir Robert S. Ball, F.R.S., need no comment:

"We were schoolfellows at Tarvin Hall, near Chester. He was there distinguished for his mathematical powers as a boy, and his excellent qualities of head and heart were well appreciated by boys and masters alike. The same school was afterwards transferred to Abbotts Grange at Chester. He was two or three classes above me, and that meant of course a wide gulf in those days. But I remember well how kindly and clearly he explained to me the phases of the moon, some fifty years ago, during a walk round the playground. In later life I saw but little of him . . . I am glad to think we always preserved the old friendship, and even decades in which we saw nothing of each other did not prevent the pleasant revival when we did meet. I was much touched by seeing in the papers in his obituary notice that I was considered a lifelong friend of his. That I certainly was for half a century and more, but at the same time I could not but reproach myself for not having striven more during all those long years to know more of one for whom I had the warmest esteem and regard."

THE SCIENTIFIC RESULTS AND AIMS OF MODERN APPLIED MECHANICS.*

By **Professor A. Sommerfeld** of Aachen. (Translated by **R. M. Milne**.)

It is very far from my purpose to construct an antithesis between a purely scientific and a technical conception of the Problem of Mechanics. The conflict which undoubtedly existed some years ago, and at times was waged with bitterness, has come to a satisfactory conclusion; and a generous appreciation of the two different directions of investigation has taken the place of unseemly strife. From my personal knowledge of the Technical

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* Professor Sommerfeld has kindly given permission to publish this abridged translation of the address, which he delivered to the Naturwissenschaftliche Hauptgruppe in Sept. 1903. The complete address is to be found in the *Physikalische Zeitschrift*, Vol. IV. No. 26*b*, pp. 773-782.

I am indebted to my colleague Mr. C. S. Jackson for suggesting the translation and for occasional advice on the matter of abridgment, and on the rendering of some passages. The whole address is worthy of the attention of any teacher of elementary mechanics, who wishes to get some idea of the more advanced methods and the recent developments of mechanical teaching in Germany.

High School in Aachen, I at least can lay stress on the fact that I have been encouraged in the most ready manner by my technical colleagues of all departments

If I were required to characterize the modern developments of Applied Mechanics, I might on the one hand point to the all-pervading demand for securing the experimental foundations of the science or on the other hand to the application of the more powerful theoretical methods

One cannot be surprised if in many branches of Applied Mechanics the experimental foundation is insecure, treated as they are at the outset by traditional rules, the application of which in the special case gives pause for thought In the first place, experimental enquiry and criticism reduces every Natural Science to what it really ought to be, an Investigation of Nature The erection of the new research laboratories in our High Schools has afforded this opportunity to the budding engineer The engineers of machine construction took the lead in the demand for experimental research. They have to day the satisfaction of possessing richly equipped laboratories for research and instruction in almost all German High Schools The requirements of structural engineering soon gave rise to a corresponding demand It is hoped that it will not be long before every High School has also its laboratory for the study of the problems presented by the employment of stone and iron in structural work In former times only few students, by reason of their specially fortunate position, were able to carry out regular experiments in the sphere of Applied Mechanics. To-day however every student has at least for six months the opportunity of seeing with his own eyes the scientific problems of the subject in experimental form

I might allude to the Theory of Earth Pressure as one branch, in which experimental research was urgently required This theory, which we have received from the hands of Coulomb, Poncelet and Rankine, were it ever so plausible, can scarcely be maintained *a priori* It transfers the laws of sliding friction, which hold good for solid dry bodies with smoothed surfaces, to the circumstances of earth with its ill defined constitution, and works with the angle of friction for the sliding of earth on earth, or of earth on walls, without adducing proof that a real meaning attaches to these ideas for the case in hand Obviously such a proof can only be obtained by experimental research this has been undertaken many times without any conclusive result It is gratifying that in the new laboratory for structural design in Charlottenburg, under the direction of Muller Breslau, this problem has been given the first place The results of the experiments, which have been planned on an extensive scale, are not completely published, we venture to hope that they will establish a firm foundation for this difficult subject

Similar difficulties arise in all those branches of Mechanics in which friction plays an essential part The mathematician and physicist are only too ready to avoid problems involving friction But for the engineer the problem of friction is the problem of his life

In my opinion the teacher of Mechanics should bear in mind that the original source of all natural knowledge is experiment No one in these days dreams of introducing the beginner in Physics and Chemistry to the subject without confirming his lessons by extensive experiments Why should not Mechanics also travel along the "royal road of experiment"? The older books on mechanics are of a strong deductive and almost dogmatic character The learner might readily get the impression that the rigid structure of laws and proofs was complete and unbroken like Euclid's Elements, as though all that was required of experiment was that it should occasionally provide a numerical constant I do not believe that this is the spirit of modern science. Rather am I of opinion that it is quite as instructive to be directed to the lack of theory as to be compelled to admire its pretended completeness. The time that is spent in instruction by way of experiments in Mechanics is richly repaid by a deepening and quickening of conception, whereby a memory picture of definite measurements and force magnitudes is added to the abstract mathematical laws. In particular I have made successful experiments with the beautiful apparatus of Topler,* which permits of the verification of all the fundamental laws on the equilibrium and the motion of bodies.

According to inquiries I have made, the Government is most ready to afford the means for putting life into instruction in Mechanics. In view of measures, which have been taken at other High Schools also, with respect to apparatus for instruction, I am of opinion that the time will soon be past when Mechanics is presented to the learner in the mathematical garb of a mere discipline in calculation and graphical work. It is to the corresponding change which in the case of Chemistry and Physics took place some fifteen years ago that these sciences owe their present vitality.

One distinctive feature of the discussion, somewhat superficial to be sure, appears when we attempt to separate the interests of the structural engineer on the one hand and the machine engineer on the other. In the earlier times mechanical theory pre-eminently served the aims of the former. For this reason Statics and Graphics constitute the foundation of Applied Mechanics. The development of engine construction and Electrotechnics has created a change in this respect. Dynamics presses forward more and more into the field of the engineer. Probably one might aver that the structural engineer employs Mechanics more extensively and the machine engineer with greater concentration. The total amount of computation and graphical work, which the construction of a bridge or roof demands, is doubtless greater in extent than the necessary mechanical principles for the design of a machine. Nevertheless in the latter lie the more abstruse problems. The elastic strains in the pieces of a machine are more various and in general more speculative (Kuhner) than those of the pieces of a structure; besides, it is here first of all that the complete system of Mechanics, that is to say Dynamics, comes into its inheritance.

Among the theoretical methods of structural engineering, the principle of Work variation takes a leading position. Just as the idea of work, by reason of the Principle of Virtual Work, is all powerful in Statics, so it gives their simplest form to the Statics of structures under elastic strains, as soon as the expression for the work of the elastic forces has been formed in its special manner. In this connection we come across the important laws of "Minimum Work-variation" and "Differential coefficients of Work variation with respect to the externally applied forces," which bear the name of the Italian engineer Castigliano.

At this point I might illustrate the "Minimum Principle" of Castigliano by the simplest possible example that lies to hand. A table with three legs is statically determined, that is to say, we can with the aid of the rules of ordinary Statics of solid bodies, determine how the applied load on the table together with its own weight is distributed among the three legs. The table with four legs is however statically undetermined. We can in fact suppose any force x we please to act in one of the four legs, and can then for the remaining three legs calculate the necessary forces, which together with x will give equilibrium with the load on the table. (Of course we suppose that the resultant of the loads does not lie in the plane of two of the legs, otherwise the proposition would be trivial.) An infinite number of equilibrium distributions would thus be possible. How then does Nature make choice amongst this infinity of force systems? To this question Castigliano's Minimum Principle gives the answer. Nature favours that choice whereby she escapes with the smallest output of change in work. It may be assumed that the table yields slightly in a uniform manner and thus we obtain the combined alteration of work in the legs of the table, which would arise from the given loads and the reactions transmitted from the floor. Our force x is then the only unknown, for we have already expressed the stresses in the other legs in terms of x and the given loads. Thus the Work-variation will be a perfectly determined function of x of the second degree. The application of the condition for a minimum leads to a linear equation for this unknown.

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* Described in the supplement to Dyck's Catalogue of mathematical models, apparatus and instruments, Munchen 1893.

I can scarcely conceive a more comprehensive and instructive method for the solution of mechanical problems. The generality is not affected if we replace our special case by any frame we please, as for example a bridge girder with redundant bars or redundant supports, or if in place of a braced system any structure subjected to elastic strains. Instead of a single linear equation we have several to determine the unknown quantities. I might here mention that the method is applicable to the determination of the displacements of the joints of a braced structure, the deflection of a beam, the torsion of a shaft, and in short to every sort of elastic strain depending on the Work Function.

As I have already had occasion to remark, Applied Dynamics was at first overshadowed by the luxuriant growth of Statical problems. To Radinger is due the great service of having awakened the dynamical conscience of the student of Applied Mechanics. He discovered afresh for machine design the Newtonian axiom, that the product of mass and acceleration is force. In reality an appreciation of this law alone suffices to make one understand, how the mass of the forward and backward moving parts of a reciprocating engine diminishes the effect of the steam in the one phase of the throw, and in the other phase assists it, and that in the so called inertia effect of the pieces of a machine, we possess a means of influencing the transmission of force in an advantageous manner. In laying stress on this new knowledge Radinger has not altogether kept himself from exaggeration and he has perhaps in consequence overrated the future of high speed engines, for which the inertia effect comes most prominently into question.

I might take this opportunity of remarking that the dynamics of a Crank, which is treated synthetically in books on Applied Mechanics, in as much as the action of the forces in the separate members is investigated consecutively, at the same time furnishes the most beautiful illustration of the general analytical methods. This example has as great a degree of generality and simplicity as can be desired. We have in this instance to deal with a system with one degree of freedom, or a constrained motion for which the impressed force depends on the position of the system (*i.e.* on the position of the piston for the time being). [The effect of the regulator, which increases the number of degrees of freedom to two, is here naturally left out of consideration.] Applying to this system the analytical expression of d'Alembert's Principle, Lagrange's General Equations, or one of the other general principles of Mechanics, we obtain quite spontaneously in the different members of the resulting equations, the precise expression for the shear in the crank pin and for the inertia effect. In addition we discover, what in the elementary treatment is obscured, that the to and fro moving parts contribute to the effect of the flywheel as if half the masses of these parts were concentrated in the crank pin.

Perhaps the more constructive treatment of the crank mechanism, which is so important in applied work, is more instructive than the general analytical method. One should guard against presenting such analytical generalities to the student as an introduction to Applied Mechanics. Indeed its extraordinary power will be brought home to the mind of the *advanced* student by taking a suitable example. The teacher of Mechanics in our universities, where the analytical methods are fully investigated as an end in themselves, should not fail to take the motion of a crank as an example.

One of the most beautiful investigations in Applied Mechanics, which has been brought to light within the last decade, is connected with the inertia effect of engines. I refer to the theory of Balancing of compound engines by Otto Schlick. In this connection we are not dealing with a mechanical speciality so much as a question of a general nature, on which every one who ponders on the subject of Mechanics should form his own conclusions. In truth, it was the stern necessity of Applied Mechanics (as is so frequently the case) that timed the solution of the problem. As in the case of locomotives, so also in ship building, the question of Balancing has made its appearance. It may be stated as follows: we are given a system of masses which move in a prescribed manner. These transfer to the frames on which they work (viz. the bed of the locomotive or the hull of the ship) the forces arising from the inertia of their motion, and in consequence the frame is partly set in motion as a whole and partly strained elastically. For locomotives only the movement as a whole comes under observation: the component effects, which result when we consider the force and couple separately, are called in railway working "Zucken und Schlingern" (hammering and rolling). In ships on the other hand it is the elastic deformations which are particularly important and dangerous, if the period of the natural vibration of the ship comes near the period of the engine stroke; if, in short, as one may say, the construction of the ship is tuned to the measure of the ship's engine. In order to divert his attention from all accidental circumstances Schlick compared the vibration of the ship to the swinging of a freely moving beam on which periodic bending forces act, and studied in his beautiful models the resonance effect between the proper swing of the beam and the induced periodic force.

The frequency of a beam or of a ship diminishes with increase in length; so it appears that with continued increase in length (in which direction modern developments tend) those limits must of necessity be reached for which the dangerous resonance effect between the period of engine stroke and natural vibration of the ship comes into play.

It appears then that ship building must either forsake the chosen course or the inertia effects must be rendered harmless. An old political principle suggested itself—to separate the enemies into two forces and pit the one against the other. The actual means employed were provided by a special arrangement of the angle of intersection of the different pistons and a special choice of the masses and space ratios of the individual working parts. In such wise the inertia forces of the ship's engine were mutually destroyed and the hull of the ship was freed from its tormentors.

Mathematically the expression for perfect balancing (to all intents and purposes it is perfect when taken to the second order) receives the form of a law expressed by eight simple equations, in the further treatment of which Schlick was assisted by Schubert and H. Lorenz. It may be mentioned that the modern steam ships Kaiser Wilhelm der Grosse and Deutschland are provided with Schlick's balancing engines, the former with the perfect Balancing of the first order and the latter with an approximate Balancing of the second order. These masterpieces of German engineering art, which form the objects of our justifiable national pride, have become through the perfection of the theory of Balancing the best of their kind.

I now approach the last problem to which we shall allude to-day: the theory of the friction of a shaft in its bearing. As to the effects of friction there are two diametrically opposed theories, the one the theory of solid friction, which originated with Coulomb and has been already alluded to in connection with earth pressure, and the other the theory of fluid friction, which in its simplest form was stated by Newton. In Applied Mechanics the former theory is so much in vogue that it has been applied to the theory of the friction of bearings, although the presence of a fluid lubricant is absolutely essential. In accordance with the theory of the friction of solids, it is usual to assume that the magnitude of the friction couple is proportional to the pressure on the axle, or, more precisely, is equal to the product of a coefficient of friction, the radius of the axle and the pressure on the bearing. The coefficient of friction might then be regarded as an empirical constant to be determined by experimenting with the bearing under a given load and a given period of revolution.

The theory of fluid friction was first applied to bearings by the Russian engineer Petroff. According to this theory, the whole loss of energy by friction occurs in the body of the lubricant, and is expended in rubbing the individual layers of the lubricant against one another, the bounding layers adhering to the revolving axle and the fixed bearing respectively. If we assume that the axle and bearing are accurately concentric, the friction couple is on this view proportional to the velocity of rotation and independent of the pressure. The coefficient of viscosity of the lubricant and the diameter of the bearing enter into the factor of proportionality. Further it was Osborne Reynolds who expounded more fully and improved the hydrodynamical theory of the friction of bearings, in as much as he abandoned the hypothesis of a central position for the axle: this new point of view was necessitated by the fact that the hydrodynamical friction and pressure should make equilibrium with the pressure on the axle communicated from without.

What then has experiment to say to the one or the other theory? Generally speaking the reply is as follows. For low velocities or heavy loads the magnitude of the pressure on the axle is the important factor. For high velocities or relatively small loads the friction couple is independent of the pressure on the axle. In the former case the theory of dry or solid friction represents the phenomena, while in the latter the action and behaviour are what might be expected according to the theory of fluid friction with a centrally situated axle.

I might refer to one point in connection with this. According to the solid friction theory, the axle will touch the bearing at a point, which is displaced from the line of action of the resultant pressure on the axle, in a sense opposite to that of the rotation. On the other hand, according to the hydrodynamical theory, the point of closest approach between axle and bearing and the point of greatest hydrodynamical pressure is displaced in the same sense as that of the rotation. At my suggestion this was put to the test in the early part of this year by Herr Becker, chief engineer in the railway works at Witten. Certain locomotives sent in for repair were examined for the wear of their bearings. Out of 20 bearings brought under inspection, it appeared that 16 had worn more in front, i.e. corresponding to the sense of rotation and in only 2 behind, while in the case of the remaining 2, the point of greatest wear was uncertain. It would seem then that the hydrodynamical theory is supported in a striking manner by this little statistical report.

These developments of Applied Mechanics, of which I have spoken, can serve no other purpose than to show that a lively scientific interest pervades the subject, that it teems with problems, is rich in difficult and elusive questions, and is laden with beautiful almost ripened fruits, which only await the skilful hand that knows how to pluck them.

The time is gone once and for all when the physicist and mathematician could superciliously hold himself aloof from the pursuits of the technical worker, because he perceived a smaller degree of scientific rigour in these branches than in his own particular sphere. Applied sciences, at least here in Germany, have developed from their own innate power a confident and self sufficient position. We the theoretical enquirers record it to our honour, if we can assist in the building up of applied science, and we appreciate our good fortune whenever it brings us into active contact with the problems of technology.

MATHEMATICAL NOTES

143 [D 2 a] *A set of criteria for convergency or divergency of series of positive terms*

It is common to regard I and the first part of III of the following more complete list as giving the fullest information on the subject which is afforded without the application of special methods, based on condensation, separation or rearrangement of terms, to particular series There is, however, a natural reluctance to accept as final anything but a full reference to all possibilities as to the value of $a_n \frac{u_n}{u_{n+1}} - a_{n+1}$ There is besides a discontent with the inequality of allusion to convergency and divergency as to convergency, I gives an absolute test, free from any use of knowledge previously acquired, whereas III gives only a test as to divergency which presupposes the possession of information about the divergency of $\Sigma \frac{1}{a_n}$ The most commonly used scale of included tests is obtained by taking for a_n in turn 1, n, $n \log n$, $n \log n \log(\log n)$, , and III enables us to say that certain series examined are divergent because $\Sigma \frac{1}{n}$, $\Sigma \frac{1}{n \log n}$, are divergent But V does not use these facts as already known It is readily applied to prove them

In the following, a, A, β, B mean assignable (and therefore finite) positive numerical constants, independent of n $\Sigma u_n = u_1 + u_2 + u_3 +$ is a series of positive terms, and a_1, a_2, a_3, a sequence of positive quantities (a_n dependent on n) assigned in any way at convenience

I *If from some value of n onwards* $a_n \frac{u_n}{u_{n+1}} - a_{n+1} > a > 0$, *then* Σu_n *is convergent*

II *If from some value of n onwards* $a_n \frac{u_n}{u_{n+1}} - a_{n+1} > 0$, *whether tending to limit 0 or not, then* Σu_n *is convergent in case* $\Sigma \frac{1}{a_n}$ *is In case, however,* $\Sigma \frac{1}{a_n}$ *is divergent,* Σu_n *is divergent provided* $a_n u_n > \beta > 0$

III *If from some value of n onwards* $a_n \frac{u_n}{u_{n+1}} - a_{n+1} < 0$, *then* Σu_n *is divergent in case* $\Sigma \frac{1}{a_n}$ *is In case, however,* $\Sigma \frac{1}{a_n}$ *is convergent,* Σu_n *is convergent provided* $a_n u_n < B < \infty$

IV *If from some value of n onwards*

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} < -a < 0, \text{ and } a_n u_n < B < \infty,$$

then Σu_n *is convergent*

V. *If from some value of n onwards*

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} < 0, \text{ but } > -A > -\infty,$$

and if $a_n u_n$ *tends to infinity with n, then* Σu_n *is divergent*

Cases in which these criteria give no information are

(1) when $a_n \frac{u_n}{u_{n+1}} - a_{n+1}$ has not always the same sign for large values of n;

(2) when $a_n\frac{u_n}{u_{n+1}} - a_{n+1} > 0$ but has limit 0, and there is doubt as to the convergency or divergency of $\Sigma\frac{1}{a_n}$;

(3) when $a_n\frac{u_n}{u_{n+1}} - a_{n+1} > 0$ but has limit 0, and a_nu_n has limit 0, and $\Sigma\frac{1}{a_n}$ is divergent;

(4) when $a_n\frac{u_n}{u_{n+1}} - a_{n+1} < 0$ but has limit 0, and $a_nu_n < B < \infty$, and there is doubt as to the convergency or divergency of $\Sigma\frac{1}{a_n}$;

(5) when $a_n\frac{u_n}{u_{n+1}} - a_{n+1}$ tends to $-\infty$, and a_nu_n tends to infinity, and $\Sigma\frac{1}{a_n}$ is convergent or doubtful.

The overlapping of certain criteria gives the following information:

(i) If $a_n\frac{u_n}{u_{n+1}} - a_{n+1} > a > 0$, and $a_nu_n > \beta > 0$, then $\Sigma\frac{1}{a_n}$ is convergent;

(ii) If $a_n\frac{u_n}{u_{n+1}} - a_{n+1} < -a < 0$, and $a_nu_n < B < \infty$, then $\Sigma\frac{1}{a_n}$ is convergent.

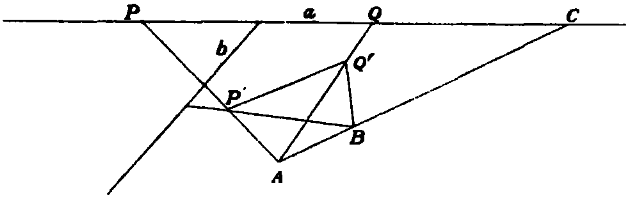
Of the five main criteria, II. and III. are only tests by comparison. They express in other terms, taking $\frac{1}{a_n}$ for c_n or d_n, that, if Σc_n is a convergent and Σd_n a divergent series of positive terms, Σu_n is convergent if from some n onwards the ratio $\frac{u_n}{c_n}$ never surpasses an assignable numerical magnitude, and on the other hand is divergent if $\frac{u_n}{d_n}$ never falls below an assignable magnitude greater than zero.

The establishment of the other tests, which have an absolute character, depends secondarily on these laws of comparison but primarily on the one fundamental fact as to convergency and divergency that $\Sigma(v_n - v_{n+1})$ is convergent or divergent according as the sequence $v_1, v_2, v_3, \ldots v_n, \ldots$ does or does not tend to a limit as n increases. If, for instance (to take IV. and V.), $v_n - v_{n+1}$ is always negative from $n = r$ onwards, so is $v_r - v_{n+1}$ by addition. No term of the sequence $v_{r+1}, v_{r+2}, \ldots$ is so small as v_r. Also the terms of the sequence increase as we go on in it, since every $v_n - v_{n+1}$ is negative. If all are less than some B, there must be a definite quantity not greater than B which separates those quantities between v_r and B which v_n can be made to exceed, by increasing n sufficiently, from those which it cannot; and the sequence $v_1, v_2, v_3, \ldots$ tends to this limit. If no such B can be assigned it tends to no limit. Thus, taking a_nu_n for v_n, there is convergency of $\Sigma(a_nu_n - a_{n+1}u_{n+1})$ under the circumstances of IV. but not under those of V. Applying then the laws of comparison to the negative of this series, regarded as Σc_n or Σd_n, and the series Σu_{n+1}, the assertions of IV. and V. follow.

E. B. Elliott.

144. [P. 3. b. α.] *To find the relation between two maps of the same contour on the stereographic projection.*

Let P' be any point of the curve σ lying on a sphere whose centre is O. Let α, β be the projections of σ from any two points A, B of the sphere on to the planes a, b through O parallel to the tangent planes at A and B. Let C be the projection of B from A on a. Rotate b about the intersection l of a and b through an angle AOB till it coincides with a. Then β is derived from a by inverting with respect to a circle of radius $CA (= OA \operatorname{cosec} \frac{1}{2}AOB)$ whose centre is C, and then reflecting in l.



For let Q' be the reflexion of P' in the plane bisecting AB at right angles, and let P, Q be the projections of P', Q' from A on a. Then

$$2OA^2 = AP . AP' = AQ . AQ',$$

and therefore the angle $QPA = P'Q'A = QAC$ (since $P'Q'$ is parallel to AB). Therefore $CP . CQ = CA^2$; and by symmetry Q is the reflexion in l of the projection of P' from B on b when the plane b has been brought to coincide with a by rotation about l. HAROLD HILTON.

Univ. Coll., Bangor.

144. [L^1. 1.] *Proofs of theorems in geometrical conic sections.*

S, S' are the foci, C the centre, A the vertex corresponding to S, and X the point where SA meets the directrix corresponding to S.

1. Given S, A, and X, find the points at which a straight line l parallel to SX meets the conic.

Draw XM, AR perpendicular to SX to meet l in M and R. Let SM meet AR in Q. Draw a circle with centre Q and radius QR. The points in which the tangents from S to this circle meet l are the points required.

The fact that a conic is symmetrical about two straight lines and many such elementary theorems follow at once from this construction.

2. Given S, A, and X, find the points at which any straight line QF meets the conic.

Draw QK parallel to SX, and XFK perpendicular to SX. Take a length D such that $D : QK = SA : AX$. Divide FS internally and externally at R and R' in the ratio $FQ : D$. The circle on RR' as diameter cuts QF in the required points.

3. From a point T tangents TY, TZ are drawn to a conic, show that the angles STY, $S'TZ$ are equal.

Draw SY, SZ perpendicular to the tangents. Bisect $S'T$ at R. Since R and C are each equidistant from Y and Z, CR is perpendicular to YZ, and therefore $S'T$ is perpendicular to YZ. Hence

$$S'TZ = 90^\circ - TZY = 90^\circ - TSY = STY.$$

4. If the tangents are at right angles, find the locus of T. Construct as in (3). Then

$$2CT^2 = ST^2 + S'T^2 - 2CS^2 = 4YR^2 + 4CR^2 - 2CS^2$$
$$= 2CY^2 + 2CZ^2 - 2CS^2 = 4CA^2 - 2CS^2.$$

Therefore T lies on a circle whose centre is C.

This method also gives the locus of the intersection of two perpendicular lines, each of which touches one of two fixed confocal conics.

5. If CP, CD are conjugate semi-diameters of an ellipse, $SP . S'P = CD^2$.

$$2SP . S'P = (SP + S'P)^2 - SP^2 - S'P^2 = 4CA^2 - 2CP^2 - 2CS^2 = 2CD^2.$$

This proof may be modified to suit the case of a hyperbola.

Univ. Coll., Bangor. HAROLD HILTON.

145. [L^2. 10. g.] *A circle and sphere connected with a confocal system of conics and a confocal system of conicoids respectively.*

I. The well-known equation for the squares of the semi-minor axes of the two conics of a confocal system which pass through any point $x'y'$ is

$$\lambda^2 - \lambda(x'^2 + y'^2 - a^2e^2) - a^2e^2y'^2 = 0. \quad \text{(See Smith's } \textit{Conics.)}$$

A geometrical theorem follows at once.

For the sum of the roots is $x'^2 + y'^2 - a^2e^2$.

Hence the sum will be positive, zero, or negative, according as the point $x'y'$ is outside, on, or inside the circle $x'^2 + y'^2 - a^2e^2 = 0$.

But $b_2{}^2$, the square of the minor axis of the hyperbola, is the negative analytical quantity, and is the negative of the square on the geometrical minor axis. Hence the circle on SS' as diameter divides those points, which are such that of the two conics through them the minor axis of the ellipse is greater than the minor axis of the hyperbola, from those points having the converse property, while for points on the circle itself the minor axes of the ellipse and hyperbola are equal

II. Again for a system of conicoids the equation that gives the squares of the least axes of the three conicoids through any point x', y', z' is

$$\frac{x'^2}{a^2 - c^2 + \lambda'} + \frac{y'^2}{b^2 - c^2 + \lambda'} + \frac{z'^2}{\lambda'} = 1.$$

Hence the sum of the roots is

$$x'^2 + y'^2 + z'^2 - (a^2 + b^2 - 2c^2),$$

and in a similar way the points, which are such that of the three conicoids of the system through them the square on the least axis of the ellipsoid is greater than the sum of the squares on the geometrical least axes (viz. the negative of the analytical squares) of the two hyperboloids, are separated from those points for which it is less by the sphere concentric with the system and passing through the angular points of the square formed by drawing parallels through each of the foci in* each of two pairs of foci of the system that lie in the plane containing the two larger axes of all the ellipsoids of the system, lines parallel to the join of the other pair; and on the sphere itself there is equality. H. L. TRACHTENBERG.

Trinity Coll., Cambridge.

146. [R. 5. a.] *To prove that the circles on one side of the radical axis of any given non-intersecting coaxial system can be described simultaneously by a swarm of particles under the attraction of a central force.*

The above theorem was devised after the converse of Hamilton's theorem on the law of force in a conical orbit had been demonstrated to me. This converse is: The possible orbits under a central force varying directly as the distance from the centre of force and inversely as the cube of the distance from a fixed straight line are conics having the centre of force and the line for pole and polar. There is one point to be noticed, viz. orbits which are integrals of the equations of motion that are cut by the line in real points must be rejected, as by reference to the law of force it is apparent that at such points the force becomes infinite and the equations of motion used are equations of motion under assumption of finite force and preclude such a case. To prove the above theorem we see that if we take the limiting point on the same side of the radical axis as centre of force, and the line parallel to the radical axis through the other limiting point for the given line, then since by a theorem in geometry all these circles have this point and line for pole and polar, and since the line does not cut any in real points, they are possible orbits.

There remained only one point to settle.

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* It will easily be seen which two of the three pairs of foci are meant.

From the equation of the orbits given me it was apparent that all conics having the centre of force and given line for pole and polar were not obtained unless we took

(1) all orbits described under an attractive force of above magnitude ;

(2) all orbits described under a repulsive force of above magnitude.

But I deduced from that equation that the circles under consideration came under case (1).

Hence the theorem that the circles can be simultaneously described under an attractive central force is true. H. L. TRACHTENBERG.

Trinity Coll., Cambridge.

147. [V. 1. a.]. *On decimalisation of money.* (Cf. Note 140, p. 383, Vol. II.)

Shillings and sixpences can be expressed accurately as decimals of a pound.

¼d. = £·001 + ·01d. accurately.

To multiply £3. 9s. 7½d. by 365.

£3. 9s. 7½d. = £3. 9s. 6d. + 6 f. = £3·475 + £·006 + ·06d.
= £3·481 + ·06d.

$$\begin{array}{rr} \pounds & \text{d.}\\ 3{\cdot}481 + & {\cdot}06\\ 365 & \\ \hline 1044{\cdot}3 & 21{\cdot}90\\ 208{\cdot}86 & \\ 17{\cdot}405 & \end{array}$$

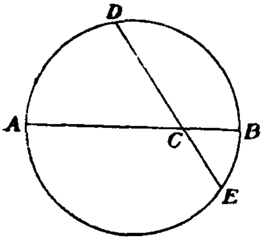
£1270·565 + 21·90d.
=(£1270. 11s. + £·015) + (1s. 9·75d. + ·15d.)
= £1270. 11s. 3¾d. + 1s. 9¾d.
= £1270. 13s. 1½d.

This work can be abbreviated. I have found it valuable as a help to accuracy (*a*) to look for a £·015 in the first product (after sixpences have been removed), (*b*) to then look for a ·15d. in the second part.

E. E. CHAMBERS.

148. [X. 4. b. α.] *A graphical solution of the typical quadratic equation* $ax^2 \pm bx + c = 0$.

(i) The fact that the rectangles contained by the segments of intersecting chords of a circle are equal leads to a very simple graphical solution of the equations of the form $ax^2 \pm bx + c = 0$ where a, b, c are all positive quantities.



(*a*) When $b^2 - 4ac$ is positive or the roots are real.

Take any two numbers p, q whose product is $\frac{c}{a}$ and sum greater than $\frac{b}{a}$.

Draw ACB such that $AC = p$ and $CB = q$.

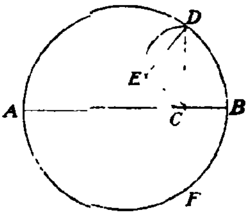
On AB as diameter describe a semicircle and through C by the ordinary geometrical method draw a chord DCE of length $\frac{b}{a}$. Then DC and CE represent the roots of $ax^2 - bx + c = 0$ and $-DC$ and $-CE$ the roots of

$$ax^2 + bx + c = 0.$$

For $DC . CE = AC . CB = p . q = \frac{c}{a}$ and $DC + CE = \frac{b}{a}$.

(β) When b^2-4ac is negative or the roots are imaginary.

As before, take any two numbers p, q whose product is $\frac{c}{a}$ and sum greater than $\frac{b}{a}$. Draw ACB such that $AC=p$ and $CB=q$. On AB as diameter describe a semicircle, and it will now be found that all chords drawn through C are greater than $\frac{b}{a}$. Hence we conclude the roots are imaginary and proceed as follows.



Through C draw the chord DCF perpendicular to AB, and on DC as diameter describe a circle. From D draw the chord DE equal to $\frac{b}{2a}$ and join EC.

The roots of $ax^2-bx+c=0$ will be $DE+\iota . EC$ and $DE-\iota . EC$ where 2 represents $\sqrt{-1}$.

Now $$ED=\frac{b}{2a} \text{ and } DC^2=AC . CB=p . q=\frac{c}{a}.$$

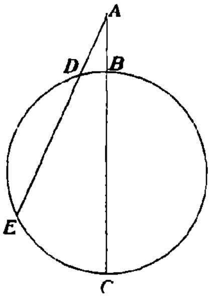
Hence $$EC^2 = DC^2 - ED^2=\frac{c}{a}-\frac{b^2}{4a^2}=\frac{4ac-b^2}{4a^2};$$

$$\therefore\ -EC^2=\frac{b^2-4ac}{4a^2};$$

$$\therefore\ \iota . EC=\frac{\sqrt{b^2-4ac}}{2a};$$

$$\therefore\ DE\pm\iota . EC=\frac{b\pm\sqrt{b^2-4ac}}{2b},$$

that is, $DE\pm\iota . EC$ represent the roots of $ax^2-bx+c=0$.



(ii) The corresponding proposition for chords intersecting outside the circle enables us to solve $ax^2\pm bx-c=0$.

Take any two numbers p, q whose product is $\frac{c}{a}$, provided their difference is greater than $\frac{b}{a}$.

Draw ABC so that $AC=p$ and $AB=q$.

On BC as diameter describe a circle and from A draw a secant ADE such that DE is equal to $\frac{b}{a}$.

Then AE and $-AD$ are the roots of $ax^2-bx-c=0$ and $-AE$ and AD are the roots of $ax^2+bx-c=0$.

For $-AE . AD=-AB . AC=-p . q=\frac{c}{a}$ and $AE-AD=DE=\frac{b}{a}$.

W. O. HEMMING.

REVIEWS.

A Treatise on the Line Complex. By C. M. JESSOP. Pp. xvi, 364. 10s. 1903. (Cam. Univ. Press.)

This is a welcome addition to the number of books in the English Language dealing with modern developments of pure mathematics. It is in fact the only book on line-geometry in English, the only other general account of this subject in English being Mr. J. H. Grace's valuable article in the *Encyclopaedia Britannica*, 10th edition, vol. 28, pp. 659-664.

In ordinary geometry of two or three dimensions points are the elements, and equations represent curves or surfaces regarded as loci of points. Again, when plane-coordinates (tangential coordinates) are employed for three-dimensional space, planes are the space-elements, and an equation represents a surface regarded as the envelope of a system of planes. So line-geometry is geometry of space in which straight lines are the elements. It is evident from the point equations of a straight line in space that four independent quantities must be specified to define a straight line; these four quantities may be treated as the coordinates of a straight line. Space therefore contains ∞^4 straight lines.

After a short introduction dealing with Double Ratio, Correspondences, etc., the author commences with the different systems of coordinates of lines. It is found advisable to use 6 homogeneous coordinates, which are connected by a quadratic identity. Plücker, who originated the subject, used the six coordinates $p_{12}, p_{13}, \ldots$, defined as follows: $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $(\beta_1, \beta_2, \beta_3, \beta_4)$ being the point-coordinates of any two points on the line, then the coordinates of this line are:

$$p_{12} = \alpha_1\beta_2 - \alpha_2\beta_1 \text{ etc., and the identity is}$$

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

Klein introduced a great improvement by a transformation of these coordinates, by which he obtained the system of coordinates x_1, x_2, x_3, x_4, x_5, and x_6, where $\Sigma x^2 = 0$. The author mainly employs the analytical method with these coordinates, the symmetry of which much simplifies the analysis. The methods of synthetic geometry however are used where they are found appropriate.

One equation in these coordinates represents a triply infinite system of straight lines, which is called a Complex. For instance, the system of all the straight lines which intersect a given straight line is a special kind of the linear complex.

Two equations represent a doubly infinite system, which is called a Congruence. The lines of a Congruence are all bitangents of some surface. If both equations are linear, the congruence consists of all the straight lines intersecting two given straight lines.

Lastly, three equations represent a singly infinite system of straight lines, that is, a Ruled Surface. If the three equations are all linear, the Ruled Surface is a quadric, or more accurately, one of the two sets of generators of a quadric (which set the author calls a Regulus); from the line-geometry point of view, the other set of generators is a distinct thing, and it is only from the point-geometry point of view that the two Reguli become one surface. As is well known, a quadric is generated by the system of straight lines which meet three given straight lines. Evidently the properties of ruled surfaces as such can best be studied by means of line-geometry.

The author deals fully with the properties of the Linear Complex and of the Quadratic Complex. There are no less than 55 distinct species of the latter complex. Numerous interesting results are also obtained relating to Congruences and Ruled Surfaces, but they are hardly capable of being summarised within the limits of a review.

The author gives a good account of the analogies between line-geometry and other four-dimensional geometries. Perhaps the most remarkable of these is the connexion, discovered by Sophus Lie, between line-geometry and sphere-geometry. Lie showed that a correspondence can be established between the two geometries, in which one sphere corresponds to one straight line, and two spheres which touch correspond to two straight lines which intersect. On this subject reference may be made to Lecture II. of Klein's charming *Lectures on Mathematics* (Macmillan, 1894). The final chapter of Mr. Jessop's book deals with partial differential equations connected with the Line Complex.

The book contains, as its title implies, a fairly complete account of the subject as at present developed, and it therefore contains much that will be of interest only to the specialist. The elements of line-geometry however are not difficult, and are of quite as much interest and importance as some of the advanced parts of ordinary Solid Geometry. A. E. Western.

The Constructive Development of Group-Theory. By B. S. Easton. Pp. iv, 89. University of Pennsylvania. 1902.

This is a bibliography and a catalogue of results, classified and tabulated. It aims at completeness, and is at any rate remarkably full and minute in its bibliographical section of 36 pages. It is the record of an examination of all the literature of Abstract and Substitution Group-Theory. The theory of linear groups is not included, nor that of continuous transformation groups. The labour expended on the production must have been great, and a very compendious work of reference has resulted. The author's prefatory note speaks of his purpose having been to construct from the material collected a continuous treatise on the subject. The result, however, is a syllabus rather than a treatise, and its atomistic character is more conspicuous than its continuity. No proofs are given, their omission being accounted for by need to economise space, and the very few literary passages—as for instance that beginning on p. 43, which speaks of the separation of abstract group-theory from that of substitution groups as not easy to mark, and proceeds to say a little more without attempting to remove the difficulty alluded to—are of slight explanatory value. The student who does not ask for exposition, but has mastered the leading ideas of the subject and wishes for guidance as to where to study more of it, will find exactly what he needs in these pages. They are richer in reference, for instance, than is the article on the subject in the *Encyclopädie der Mathematischen Wissenschaften.* The compilation seems to be the outcome of academic encouragement and of sustained effort on the part of a young researcher, and does credit both to him and to the University of Pennsylvania.

Les Mécanismes: Traité élémentaire de cinématique appliquée. H. Leblanc. Paris: Garnier Frères, 1903. Pp. 432.

Professor Huxley in the autobiographical sketch prefixed to his essays remarks that he always considered himself as a mechanical engineer *in partibus infidelium.* Inverting this idea it may be suggested that the mental satisfaction derived from the study of mechanisms is akin to that of the zoologist, for it arises from the contemplation of the almost infinite variety of results attained by variations on a very limited number of primitive forms.

For example, we recognise in the mechanism for enabling a tricycle, car, or traction engine, to turn a corner, and in certain two-speed gears and dynamometers, a combination of two pairs of co-axial bevel wheels whose axes are at right angles, which is known as the differential bevel gear. This was first employed in cotton spinning machinery. If thread, delivered at a uniform rate, is wound on to a bobbin which revolves at a uniform rate trouble will arise, because the diameter of the bobbin increases as the thread winds on to it. It was to obviate this difficulty that the differential gear, which has since played so many parts, was invented.

M. Leblanc's book should prove of great service to anyone who takes a pleasure in following out such a comparison. He has brought together some 200 mechanisms, comprising examples of brakes, safety clutches, friction clutches, change speed gears, steering gears, and free-wheel mechanisms, each illustrated by a diagram and accompanied by a concise yet wonderfully clear description.

No student of machinery can fail to derive the satisfaction alluded to, or to obtain several interesting specimens worthy of close examination, from the collection of M. Leblanc.

C. S. Jackson.

Recueil d'Exercices sur le Calcul Infinitésimal. By M. F. Frenet. Sixth Edition. 8 frcs. (Gauthier-Villars.)

The first edition of this collection of examples and their solutions appeared in 1856 and consisted of but 220 pages. It was constructed avowedly on the lines of a book well known in those days and even now of interest from the historical point of view—*Gregory's Examples.* Being intended primarily for the use of the candidates for the Ecole Polytechnique and Ecole Normale it has considerably grown with the steady expansion of the syllabus of the examinations at those institutions. It now consists of over 500 pages and contains about 120 pages more than appeared in the last edition. This is largely due to the inclusion of the theory of elliptic functions and the theory of the functions of a variable imaginary among the new subjects of the official programme. Questions dealing with residues, elliptic functions, partial differential equations, etc., have been added by M. H. Laurent, and the whole has been thoroughly revised and brought up to date.

Notions de Mathématiques. By Jules Tannery. **Notions Historiques.** By Paul Tannery. Pp. x, 352. 1903. (Delagrave.)

This little book is an endeavour to meet the difficulty caused by the extension of the course of mathematics taken by the "classe de Philosophie." The additions to that course are such as have hitherto been read by students giving a considerable portion of their time to analysis and geometry. The author aims to present the subject in such a general way as to make intelligible to the student of philosophy the methods and processes of mathematics, so that he may find that in a few years' time the language of the science is not to him an unknown tongue, and so that he will be the better able to grasp an idea of scientific progress and the applications of science which are, as the author says in his preface, "tending day by day to modify more and more profoundly our modes of life and thought." The book may therefore be considered as an introduction to an "étude réfléchie" of the subject. It gives the student an idea of the field covered by the science, the nature of the problems which it attacks and solves, the nature of a function, its variations and its graph, the mutual relationship of algebra and geometry, the relations of number and space, and so forth. One third of the book is devoted to the elements of the subject. We cannot help thinking that most teachers will learn something from a volume written with the aim that is carefully laid down in the "conseils généraux" given in the programme for the "classe de Philosophie." "The teacher must not forget that the pupils are not accustomed to the study of the subject. He must therefore avoid all abstract theory; he will not bring general ideas before his classes, but must utilise particular examples in order to throw general ideas into relief. The syllabus is to a large extent elastic, and the master may extend its scope here and there according to the aptitude of his pupils and the interest he has succeeded in exciting in them. This refers in particular to such applied mathematics as occurs in the syllabus, and this branch of the subject should be treated broadly and without too much rigour. Historical sketches should form part of the teaching. For instance, the pupils should be told how Euclid, Archimedes, etc., made use of the method of exhaustion, and they should have some notion of how and when the differential and integral calculus came into being. The object of the teacher should be to contribute to the philosophical development of the pupils, by instilling into them important ideas; and on the other hand he must furnish them with the concepts which they will need should they ever require to take their certificates in physics, chemistry, or any other form of natural science." With these objects in view it cannot be that the book is without interest to the British teacher of Mathematics. We only add that the 25 pages given by M. P. Tannery to his historical sketch are all too short.

A First Course in Infinitesimal Calculus. By D. A. Murray. Pp. xvii, 439. 1903. (Longmans, Green.)

Professor Murray has written an introduction to the infinitesimal calculus for those whom circumstances compel to study without the guidance and assistance that a teacher can afford. We have no hesitation in saying that it is admirably adapted for this as well as for ordinary school purposes. The author is evidently a teacher of great experience, and is familiar with the special pitfalls which beset the path of the ordinary student in approaching this subject. The best teacher is he who can anticipate the difficulties and smooth the path of the student without however removing every obstacle. The sooner the student learns that mathematics is a hard subject the better. He must be prepared to find that there is much to be grappled with, and it is by boldly facing the difficulties and trying to understand them without undue assistance that the intellectual fibre is strengthened, and the study of mathematics becomes a mental training of incomparable value. In his preface, Professor Murray takes the unusual course of expressing his obligations to "many students whose difficulties and original opinions have interested and stimulated me." But the author has not made the mistake of making things too easy for his classes. And for the sake of those whose interest in the development of the various sections has been excited he has supplied copious references to the best text-books in England and America. While there is nothing novel in the content of the book, the author is to be congratulated on the skill with which the material is arranged, and the careful manner in which the pupil is led on from stage to stage of the subject.

Mathematical and Physical Papers. By the late Sir G. G. STOKES, Bart. Vol. IV. Pp. viii, 378. 1904. (Cam. Univ. Press.)

The fourth volume of these papers includes those published between 1853 and 1876. They have been carefully selected by Professor Larmor with the assistance of Lords Kelvin and Rayleigh and Mr. F. G. Hopkins, "only those memoirs and notes involving distinct additions to scientific knowledge being included." Among the mathematical papers are "On the Discontinuity of Arbitrary Constants which appear in Divergent Developments," with a supplementary paper; "On the Internal Distribution of Matter which shall produce a given Potential at the surface of a Gravitating Mass"; "On the Communication of Vibration from a Vibrating Body to a surrounding Gas." The beautiful portrait prefixed to the volume is taken from Mr. Dickinson's painting in Pembroke College. The next volume, which is in the hands of Lord Rayleigh, will contain the remaining papers, together with a biographical notice, which will be looked forward to with great interest. A glance at the contents of this volume is sufficient to show the many-sided versatility of this great *savant*. The account of his life and work could not be placed in better hands than in those of Lord Rayleigh.

Lehrbuch der Analytischen Geometrie. Bearbeitet von O. FORT und O. SCHLÖMILCH. Erster Teil: **Analytische Geometrie der Ebene** Von. O. FORT. Siebente Auflage. Besorgt von R. HEGER. Pp. xvii, 268. 1904. (Teubner.)

This volume has changed but little since its first appearance nearly half a century ago. It is primarily intended for the students at the Konigl. Sachs. Polytechnikum at Dresden, and rather more attention than usual is paid to questions that are capable of graphical interpretation or application. For instance, we find an account of the generation of the conics by the intersection of projective pencils. Conics are treated at first individually and in detail so as to bring out particular geometrical properties; then, starting with the equation of the second degree, they are discussed and reduced to their simplest forms and incidentally, as it were, we are introduced to the general properties of the curves of the second and higher orders. The proofs are well arranged and simply put. There is no direct reference to the ideas of infinitesimal analysis. The book is beautifully printed. It concludes with the treatment of Cissoid, Lemniscate, Spirals, etc.

Précis d'Algèbre et de Trigonometrie à l'usage des élèves de Mathématiques Spéciales. By G. PAPELIER. Pp. 354. 1903. (Nony.)

This "precis" is a masterly arrangement of exactly what is necessary for the examinations of the students for whom it is intended. Although it gives the general impression of being the irreducible minimum, it is surprising to find what a wide extent of ground is covered. M. Papelier writes in a clear and attractive style.

First Lessons in Observational Geometry. By Mrs. W. N. SHAW. Pp. x, 148. 1903. (Longmans, Green.)

This admirable little book is the outcome of the conviction held by its accomplished authoress that the want of aptitude which we often think is shown by pupils for the study of geometry is more apparent than real. Failure is due to the uninviting manner in which the subject has been presented at the beginning. The essential object in the teaching of geometry is that the student shall eventually gain clear perceptions of the relationships of space. The students who can write out any amount of bookwork with the ease that is born of a good memory, quick handwriting, and constant practice, may nevertheless know very little of geometry at the conclusion of their studies, and it may be taken for granted that they will never of their own accord be induced to take up the subject again. "I found that even students who had reached a relatively high standard of success in mathematical examinations were still very uncertain about geometrical facts." Could there be any more biting criticism of the system of teaching which those examinations represent? Although we do not greatly care for the way in which Mrs. Shaw has chosen to tell her tale—in a series of conversations with Jane and William, etc.—we have nothing but admiration for the exceedingly clear and simple manner in which she develops the subject.

Grundlagen der Geometrie. By D. HILBERT. 2nd Edition. Pp. 175. 1903. (Teubner.)

The second edition of Prof. Hilbert's epoch-making work on the foundations of Geometry has doubled in size owing to the inclusion of several papers from the Math. Ann. and the Proceedings of the London Mathematical, and Transactions of the American Mathematical Society.

Introduction to Quaternions. By the late Professors P. KELLAND and P. G. TAIT. 3rd Edition. Prepared by C. G. Knott. Pp. 208. 1904. (Macmillan.)

Dr. Knott has greatly improved on the second edition of Kelland and Tait's well-known introduction to Quaternions. He has swept away the hints and solutions which were placed at the end of that edition, and he has either deleted altogether or placed among the examples the simpler geometrical illustrations which are nowadays but child's play to the well prepared candidate for University Scholarships. Chapters III. and IV. are new. The quaternion is defined as the complex number which measures the ratio of two vectors, and the associative law is assumed to hold good in product combinations. From these two principles Dr. Knott claims that the whole of Hamilton's vector algebra evolves, and moreover that O'Brien's difficulty as to the identification of vector and quadrantal versor is removed by this method of presentation. The extension of vector ideas and notations into text-books on mechanics and mathematical physics in general has led to the inclusion of a section on dynamical applications in Chap. IX. The last four articles of the same chapter introduce the student to the mysterious Nabla. Dr. Knott was a pupil of both the authors, and a colleague and friend of Professor Tait. We need only add that the work of preparing a new edition could hardly have been entrusted to more competent hands. The book is certainly the most attractive introduction to a fascinating subject. We could have wished that Dr. Knott had seen his way to give the reader some account of the history of the subject, including some information about the genius to whom we are indebted for its invention, and the steps by which he was led to the same. With the exception of this complaint we have only to congratulate Dr. Knott on the way in which his labour of love has been performed. Looking at the subject from the point of view of the teacher, we have always been struck by the plea advanced by Kelland in his preface to the first edition. The subject of Quaternions "belongs to first principles and is their crowning and completion. It brings those principles face to face with operations, and thus not only satisfies the student of the mutual dependence of the two, but tends to carry him back to a clear apprehension of what he had probably failed to appreciate in the subordinate sciences. Besides, there is no branch of mathematics in which results of such wide variety are deduced by one uniform process . . . and what is of the utmost importance in an educational point of view, the reader . . . does not require to encumber his memory with a host of conclusions already arrived at in order to advance. Every problem is more or less self-contained." And again, it is worth while quoting Tait's remarks in the preface to the second edition of his elementary treatise. "It must always be remembered that Cartesian methods are mere particular cases of Quaternions, where most of the distinctive features have disappeared, and that, when in the treatment of any particular questions scalars have to be adopted, the Quaternion solution becomes identical with the Cartesian one. . . . It appears to me that the study of a mathematical subject like Quaternions, which, while of immense power and comprehensiveness, is of extraordinary simplicity, and yet requires constant thought in application, would also be of great benefit. With it there can be no 'shut your eyes and write down your equations,' for mechanical dexterity of analysis is certain to lead at once to error. . . ."

Five-Figure Tables of Mathematical Functions. By J. B. DALE. Pp. vi, 92. 1903. (Arnold.)

Mr. Dale's set of tables will be found of service to students of applied mathematics and physics. In addition to the ordinary tables of logarithms, natural trigonometrical functions and the like, we find the following:—A table of zonal surface harmonics giving the first seven harmonics for values of the argument x at intervals of 0·01; tables of Bessel Functions giving $J_n(x)$ and $I_n(x)=\iota^{-n}J_n(\iota x)$; the values of $\log_{10}\Gamma(x)=\log_{10}\int_0^{\infty} e^{-t}t^{x-1}dx$ for values of x between 1 and 2, and $\log\Gamma(x+1)$ from $x=1$ to $x=100$; logs of factorials; and binomial coefficients for interpolation by differences. The collection will hold its own among the others in the market.

Methodisches Lehrbuch der Elementar-Mathematik. By G. HOLZMULLER. Vol. III. 2nd Edition. Pp. xiv, 370. 1903. (Teubner.)

The sections into which this book is divided are concerned respectively with Higher Geometry, Stereometry, Spherical Trigonometry, Elementary Algebraic Analysis, and equations of the third, fourth, and nth degrees. An important feature is the number of applications to questions in physics, astronomy, geodesy, navigation, steam, etc., and for this reason it is worth the attention of the enterprising teacher.

Der Geometrische Vorkursus in Schulgemaszer Darstellung. By E. WIENECKE. Pp. 97. 1904. (Teubner.)

This is an excellent introduction to experimental geometry. There is a full course of *vivâ voce* questions which should help to sustain interest and to enable the student to form sound ideas.

Formulaire de Mathématiques Spéciales. By G. PAPELIER. Pp. 217. N.p. 1904. (Vuibert & Nony.)

This is a handy little volume in cloth, interleaved with blank sheets. The ground covered includes Algebra, Trigonometry and Analytical Geometry of two and three dimensions, with the applications of the calculus. Perhaps the simplest way of shewing the reader what he may expect to find in the book is to give a random specimen:

Equations générales de Coniques (p. 103).

Equations ponctuelles generales de coniques:
Coniques passant par les points A, B, C, D. (AB, $P=0$; CD, $Q=0$).
$$f+\lambda PQ=0.$$
Coniques tangentes en A à la conique $f=0$ et passant par les points B, C.
$$f+\lambda PQ=0.$$
Coniques osculatrices en A a la conique $f=0$ et passant par le point B.
$$f+\lambda PQ=0.$$
Coniques surosculatrices en A à la conique $f=0$. $f+\lambda P^2=0$.
Coniques bitangentes en A et B a la conique $f=0$. $f+\lambda P^2=0$.
Coniques homothetiques a la conique $f=0$ et passant par les points A et B.
$$f+\lambda P=0.$$
Coniques homothétiques et concentriques a la conique $f=0$. $f+\lambda=0$.
Coniques circonscrites au quadrilatere $ABCD$ (AB, $P=0$; BC, $R=0$; CD, $Q=0$; DA, $S=0$). $PQ+\lambda RS=0$.
Coniques circonscrites au triangle ABC et tangentes en A à la droite AT ($S=0$).
$$PS+\lambda QR=0.$$
Coniques tangentes en B et C aux droites AB, AC. $QR+\lambda P^2=0$.

Coniques circonscrites au triangle ABC. $\Sigma\lambda QR=0$ ou $\Sigma\frac{\lambda}{P}=0$.

Coniques inscrites au triangle ABC. $\sqrt{\lambda P}\pm\sqrt{\mu Q}+\sqrt{\nu R}=0$.
Coniques conjuguées par rapport au triangle ABC. $\Sigma\lambda P^2=0$.
Coniques bitangentes aux deux coniques $f=0$, $f_1=0$, $\mu^2P^2+2\mu(\lambda_1 f_1-f)+Q^2=0$.
λ_1 désigne une racine de l'equation en λ, et P, Q deux fonctions linéaires correspondantes, telles que $f+\lambda_1 f_1\equiv PQ$. [illegible] a trois séries de coniques relatives aux racines de l'équation en λ.

Coniques inscrites dans une quadrilatère, $\frac{P^2}{\lambda}+\frac{Q^2}{1-\lambda}-R^2=0$; les équations des côtés étant $P\pm Q\pm R=0$, et celles des diagonales $P=0$, $Q=0$, $R=0$.

SOLUTIONS.

422. [K. 1. a.] *Three collinear points are at distances a_1, a_2, a_3 from one another and r_1, r_2, r_3 from a fourth point distant p from their line; prove that*

$$\Sigma a = 0, \quad \Sigma ar^2 + \Pi a = 0, \quad p^2\Pi a^2 = 4s\Pi(s - ar),$$

where $2s = \Sigma ar$. R. W. H. T. **Hudson.**

Solution by W. F. **Beard.**

Let x_1, x_2, x_3 be distances of the three collinear points from the foot of the perpendicular from the fourth point on the line, so that

$$a_1 = x_2 - x_3, \quad a_2 = x_3 - x_1, \quad a_3 = x_1 - x_2.$$

(1) $\Sigma a = 0$, obviously.

(2) $r_1^2 = p^2 + x_1^2$, etc.,

$$\Sigma ar^2 = p^2\Sigma a + \Sigma x_1^2(x_2 - x_3)$$

$$= -\Pi(x_2 - x_3) = -\Pi a,$$

$$\Sigma ar^2 + \Pi a = 0.$$

(3) $16s(s - a_1r_1)(s - a_2r_2)(s - a_3r_3)$

$$= -\Sigma a^4r^4 + 2\Sigma a_2^2r_2^2a_3^2r_3^2$$

$$= -\Sigma a_1^4(p^2 + x_1^2)^2 + 2\Sigma a_2^2a_3^2(p^2 + x_2^2)(p^2 + x_3^2)$$

$$= -p^4\{\Sigma a_1^4 - 2\Sigma a_2^2a_3^2\} - 2p^2\{\Sigma a_1^4x_1^2 - \Sigma a_2^2a_3^2(x_2^2 + x_3^2)\}$$

$$- \Sigma a_1^4x_1^4 + 2\Sigma a_2^2a_3^2x_2^2x_3^2.$$

Coefficient of $-p^4 = (a_1 + a_2 + a_3)(-a_1 + a_2 + a_3)(\quad)(\quad) = 0$, from (1).

Coefficient of $2p^2 = \Sigma a_1^2x_1^2(a_1^2 - a_2^2 - a_3^2)$

$$= -\Sigma 2a_1^2a_2a_3x_1^2 \qquad a_1^2 = a_2^2 + a_3^2 + 2a_2a_3$$

$$= -2a_1a_2a_3\Sigma a_1x_1^2$$

$$= -2a_1a_2a_3\Sigma x_1^2(x_2 - x_3)$$

$$= 2a_1^2a_2^2a_3^2, \text{ as in (2).}$$

Last term $= 16s'\Pi(s' - ax)$, where $2s' = a_1x_1 + a_2x_2 + a_3x_3$;

$$2s' = \Sigma x_1(x_2 - x_3) = 0;$$

last term $= 0$.

Thus $16s\Pi(s - ar) = 4p^2\Pi a^2$;

$$p^2\Pi a^2 = 4s\Pi(s - ar).$$

423. [B. 1. a.] *Evaluate in terms of the zeros of $ax^2 + bx + c$, or otherwise, the determinant of the n^{th} order,*

$$D_n = \begin{vmatrix} b+d & c+d & d & d & \dots & d \\ a+d & b+d & c+d & d & \dots & d \\ d & a+d & b+d & c+d & \dots & d \\ \dots & \dots & \dots & \dots & \dots & .. \end{vmatrix},$$

in which each diagonal is composed of the same elements and all the elements outside the central three diagonals are the same.

Shew also that the determinant is the coefficient of x^n in the expansion of

$$\frac{1}{(1 - bx + acx^2)} + \frac{dx(1 - acx^2)}{(1 + ax)(1 + cx)(1 - bx + acx^2)^2}.$$

F. S. **Macaulay.**

Solution by PROPOSER.

Let α, β be the roots of the equation $ax^2+bx+c=0$, add up the elements of D by columns, giving for the first row

$$e-c,\ e,\ e,\ \ldots e,\ e_1-a, \text{ where } e = a+b+c+nd$$

Multiply the rows by $1, \alpha, \ldots \alpha^{n-1}$, and add, giving for the first row

$$e_1-\frac{c}{\alpha},\ e_1,\ e_1,\ \ldots e_1,\ e_1-a\alpha^n, \text{ where } e_1 = d(1+\alpha+\ldots+\alpha^{n-1})$$

Similarly we can obtain for the first row

$$e_2-\frac{c}{\beta},\ e_2,\ e_2,\ \ldots e_2,\ e_2-a\beta^n, \text{ where } e_2 = d(1+\beta+\ldots+\beta^{n-1})$$

Choose multipliers λ, μ, ν for these three rows, such that

$$\lambda+\mu\alpha^n+\nu\beta^n = 0,$$
$$\lambda e+\mu e_1+\nu e_2 = 0$$

We thus make all the elements of the first row zero, with the exception of the first element, which becomes $-c\left(\lambda+\frac{\mu}{\alpha}+\frac{\nu}{\beta}\right)$ while the determinant D becomes multiplied by $\lambda+\mu+\nu$

Hence $$(\lambda+\mu+\nu)D_n = -c\left(\lambda+\frac{\mu}{\alpha}+\frac{\nu}{\beta}\right)D_{n-1},$$

$$\begin{vmatrix}1&1&1\\1&\alpha^n&\beta^n\\e&e_1&e_2\end{vmatrix}D_n = -c\begin{vmatrix}1&\frac{1}{\alpha}&\frac{1}{\beta}\\1&\alpha^n&\beta^n\\e&e_1&e_2\end{vmatrix}D_{n-1} = -\frac{c}{a}\begin{vmatrix}1&1&1\\1&\alpha^{n+1}&\beta^{n+1}\\e&\alpha e_1&\beta e_2\end{vmatrix}D_{n-1}$$

$$= -\frac{c}{a}\begin{vmatrix}1&1&1\\1&\alpha^{n+1}&\beta^{n+1}\\e+d&\alpha e_1+d&\beta e_2+d\end{vmatrix}D_{n-1} = -\frac{c}{a}\begin{vmatrix}1&1&1\\1&\alpha^{n+1}&\beta^{n+1}\\e_{n+1}&e_{1,n+1}&e_{2,n+1}\end{vmatrix}D_{n-1}$$

Hence

$$\frac{D_n}{\left(-\frac{c}{a}\right)^{n+1}\begin{vmatrix}1&1&1\\1&\alpha^{n+1}&\beta^{n+1}\\e_{n+1}&e_{1,n+1}&e_{2,n+1}\end{vmatrix}} = \frac{D_0}{\left(-\frac{c}{a}\right)\begin{vmatrix}1&1&1\\1&\alpha&\beta\\e&e&e\end{vmatrix}} = \frac{1}{a(a+b+c)(\alpha-\beta)}$$

$$D_n = \frac{\left(-\frac{c}{a}\right)^n}{(a+b+c)(\alpha-\beta)}\begin{vmatrix}1&1&1\\1&\alpha^{n+1}&\beta^{n+1}\\e_{n+1}&e_{1,n+1}&e_{2,n+1}\end{vmatrix}$$

429 [**R 4 a**] *Three equal smooth spheres of radius r rest in a hollow hemisphere of radius R with their centres in the same horizontal plane, a cone whose weight is equal to the weight of a sphere and whose semivertical angle is α, is inserted symmetrically between the spheres with its vertex downwards, prove that the spheres will separate if*

$$\cot\alpha > \frac{8r}{\sqrt{3R^2-6Rr-r^2}}$$ Pemb (c), 1893

Solution by W F BEARD

Let R be the pressure between each sphere and the bowl
„ R_1 „ „ , „ and the cone
„ R_2 „ „ „ each pair of spheres.

Let O_1, O_2, O_3 be the centres of the three spheres; O the point in which the axis of the cone meets the plane $O_1O_2O_3$.

Then $\qquad O_1O_2O_3$ is an equilateral $\triangle$ with O as its c.g.

Hence
$$O_1O=\frac{2r}{\sqrt{3}}.$$

Also, if H is the centre of the hemisphere $O_1H=R-r$.

Let $O\hat{O}_1H=\beta$. Then $\cos\beta=\dfrac{2r}{(R-r)\sqrt{3}}$; $\quad\therefore\ \sin\beta=\dfrac{\sqrt{3R^2-6Rr-r^2}}{(R-r)\sqrt{3}}$. ....(i)

The resultant of the two forces R_2 acting on the sphere centre O_1
$$=2R_2\cos 30^\circ=R_2\sqrt{3}\text{ along }OO_1.$$

Also the force R on this sphere acts along O_1H.

Plainly also the force R_1 acts in the plane OO_1H through O_1, making an $\angle\,\alpha$ with the horizon.

Let W be the weight of each sphere and of the cone.

Consider the equilibrium of the cone.

Resolving vertically, $3R_1\sin\alpha=W$ ....(ii)

Consider the equilibrium of the sphere, centre O_1.

Resolving vertically, $\quad R\sin\beta=W+R_1\sin\alpha=\dfrac{4W}{3}$ from (ii). ....(iii)

Resolving horizontally, $\quad R\cos\beta=R_2\sqrt{3}+R_1\cos\alpha$.

The spheres will separate if R_2 becomes negative or zero,

$$\begin{aligned}
&\textit{i.e.}\quad\text{if } R_1\cos\alpha>R\cos\beta,\\
&\qquad\text{if } \frac{W}{3}\cot\alpha>\frac{4W}{3}\cot\beta,\text{ from (ii) and (iii).}\\
&\qquad\cot\alpha>4\cot\beta\\
&\qquad\qquad>\frac{8r}{\sqrt{3R^2-6Rr-r^2}},\text{ from (i).}
\end{aligned}$$

442. [L^1. 1. b.] [*Mechanical Construction of a Hyperbola.*] *A triangle slides with its base on a fixed straight line: prove that the line joining a fixed point to a carried point cuts a side in a point whose locus is a hyperbola; and find the asymptotes.* R. W. H. T. Hudson.

Solution by A. W. Poole.

We may suppose the fixed point on the fixed straight line. Let the $\triangle ABC$ slide along the axis of x. Let O be the fixed point and L the carried point.

Let the coordinates of L be $\qquad l+b,\ a,$

and the coordinates of A be $\qquad l+d,\ c,$

where $l=OB$ and is variable.

Equation of LO is $\dfrac{x}{l+b}=\dfrac{y}{a}$; $\quad$ equation of AB is $\dfrac{x-l}{d}=\dfrac{y}{c}$;

$\therefore$ eliminating l we have
$$x+b=\frac{ax}{y}+\frac{dy}{c};$$
or
$$dy^2-cxy+acx-bcy=0,$$
or
$$(y-a)(dy-cx+ad-bc)+a(ad-bc)=0;$$
an hyperbola passing through O and whose asymptotes are
$$y=a,\quad d(y+a)=c(x+b);$$
$y=a$ passes through L and is parallel to OX; $d(y+a)=c(x+b)$ is parallel to AB and passes through $x=-b$, $y=-a$.

^{1}. 10. d.] *PQ is a chord of a parabola meeting the axis in a fixed o that $AH=8AS$, and $P'Q'$ its projection on the tangent at the vertex: the locus of the intersection of PQ'. $P'Q$ is a circle.* E. N. Barisien.

Solution by J. M. Child.

If the coordinates of P, Q be $(ah^2, 2ah)$, $(ak^2, 2ak)$, the coordinates of point intersection of $P'Q$, PQ' are

$$x=\frac{ah^2k^2}{h^2+k^2},$$

$$y=\frac{2ahk(h+k)}{h^2+k^2};$$

hilst condition that PQ passes through $(0, 8a)$ is

$$hk+8=0.$$

Eliminating h, k we have

$$x^2+y^2-4ax=0,$$

circle on AK as diameter, where K is the middle point of AH.

483. [L^1. 4. c.] *The tangents to an ellipse at Q and R intersect at a point P a coaxial ellipse whose axes are respectively n times those of the former curve. ind the locus of the centroid of the triangle PQR.* A. F. Van der Heyden.

Solution by W. H. Salmon.

Let CP meet the original ellipse in p and QR in V. Then if G be the ntroid of PQR,

$$VG=\tfrac{1}{3}VP \text{ and } CP=n\,.\,Cp;$$

$$\therefore\ CG-CV=\tfrac{1}{3}(CP-CV)$$

nd

$$CG=\tfrac{1}{3}(CP+2CV)=\tfrac{1}{3}\left(CP+\frac{2Cp^2}{CP}\right)$$

$$=\tfrac{1}{3}\left(n+\frac{2}{n}\right)Cp;$$

. locus of G is a coaxial ellipse, whose axes are $\tfrac{1}{3}\left(n+\frac{2}{n}\right)$ times those of the iginal ellipse.

OLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) For Sale.

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 874 to Nov. 1882. Vols. I.-IX. Edited and Published by E. Hendricks, M.A., es Moines, Iowa, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical isitor.* 1879-1881. Edited by Artemas Martin, M.A. (Erie, Pa.)]

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil ar, and not resumed). Edited by J. D. Runkle, A.M.

Proceedings of the London Mathematical Society. First series, complete. Vols. 1-35. Bound in 27 vols. Half calf. £25.

Cayley's Mathematical Works. Complete, equal to new, £10. Apply, Professor of Mathematics, University College, Bangor.

(2) Wanted.

The Messenger of Mathematics. Vols. 2, 15-20, 24, 25.

Tortolini's Annali. Vol. I. (1850), or any one of the first eight parts of the Volume.

Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

(3) Dr. Muir, The Education Office, Cape Town, will give Vol. 109, *Crelle's Journal*, to any member of the Mathematical Association whose set is without it.

BOOKS, ETC., RECEIVED.

Les Mécanismes. By H. LEBLANC. Pp. 432. 1904. (Garnier.)

Leçons de Trigonométrie rectiligne. By C. BOURLET. Pp. xii, 322. 6 frs. 1904. (Armand Colin.)

Leçons sur l'Intégration et la Recherche des Fonctions primitives. By H. LEBESGUE. Pp. viii., 136. 3 fr. 50. 1904. (Gauthier-Villars.)

Leçons de Mécanique. By E. COMBETTE and J. GIROD. Pp. 321. 3 fr. 50. 1904. (Alcan.)

The Message of Non-Euclidean Geometry. By G. B. HALSTED. (Science, March 11, 1904.)

On Generalized Space Differentiation of the Second Order. By B. O. PIERCE. Pp. 377-386. (Proceedings of Amer. Acad. of Arts and Sciences.) Feb. 1904.

On the Transfinite Cardinal Numbers of Number Classes in General. By P. E. JOURDAIN. Pp. 294-303. (Phil. Mag., Mar. 1904.)

The Message of Non-Euclidean Geometry. By G. B. HALSTED. Pp. 401-413. (Science, Mar. 11, 1904.)

Elementary and Intermediate Algebra. By J. LIGHTFOOT. Pp. viii, 422; xlii 2nd Edition. 1904. 4s. 6d. (Ralph, Holland & Co.)

Bibliography of Quaternions and Allied Systems of Mathematics. By A. MACFARLANE. Pp. 86. N.p. 1904. (Dublin Univ. Press.)

Obras sobre Mathematica. By F. GOMES TEIXEIRA. Published "por ordem do Governo Portuguès." Vol. I. Pp. viii, 402. 1904. (University of Coimbra.)

Notions de Mathématiques. By JULES TANNERY. *Notions Historiques.* By PAUL TANNERY. Pp. x, 352. 1904. (Delagrave.)

History of European Thought in the Nineteenth Century. By J. T. MERZ. Vol. II. [Chaps. VI.-XII. On Views of Nature:—Kinetic or mechanical, physical, morphological, genetic, vitalistic, psycho-physical, statistical. Chap. XIII. On the Development of Mathematical Thought during the 19th Century.] Pp. xiii, 807. 15s. net. 1903. (Blackwood.)

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NOTICE

THIRD INTERNATIONAL CONGRESS OF MATHEMATICIANS AT HEIDELBERG, AUGUST, 1904.

MEMBERS of the Mathematical Association are invited to be present at the Third International Congress of Mathematicians, which will take place at Heidelberg from Monday, August 8th, to Saturday, August 13th, 1904. The proceedings will commence with a reception at 8 p.m. on August 8th; there will be three general meetings and also meetings of six sections, viz., (1) Arithmetic and Algebra, (2) Analysis, (3) Geometry, (4) Applied Mathematics, (5) History of Mathematics, (6) Pedagogy.

The centenary of the birth of C. G. J. Jacobi will be celebrated during the Congress.

There will be an exhibition of Mathematical Models (historical and modern), and an exhibition of the more important mathematical publications during the past ten years, to which contributions on loan are invited. Intending contributors to the exhibition of literature are requested to correspond with Dr. A. Gutzmer, Professor an der Universität Jena. Social meetings, a banquet, and an illumination of the Castle will also take place.

The price of a ticket for the Congress is 20 marks; additional tickets may be obtained by members of the Congress for their own party at 10 marks each person, which are available for the general meetings and festivities.

Further information concerning the Congress may be had from Professor A. Krazer, Karlsruhe i. B., Westendstrasse 57.

REPORT OF THE M.A. COMMITTEE ON THE TEACHING OF ELEMENTARY MECHANICS.

THE Committee make the following suggestions:—

A. Preliminary Experimental Work.

1. That a simple course of experiment would be useful with a view to illustrating
   - (α) Composition and resolution of forces.
   - (β) The principle of the lever.
   - (γ) Friction.

2. The experiments should, if possible, aim at discovery, and be quantitative—mere verification is less useful.

B. General Remarks on Examples and Methods.

3. Examples should at first, as a rule, be numerical, and any of a general character should conclude with a numerical application.
4. Examples should, as far as possible, be of a practical nature.
5. A specially instructive class of example consists in compiling a table or drawing a graph to show the effect on a result of variation in a certain datum.
6. Stress should be laid on the great importance of checking results by an independent method; in particular, questions should often be worked out both graphically and by calculation.
7. Prominence should be given to geometrical methods.
8. Pupils should always be required to specify the body whose equilibrium or motion is being considered, and to indicate the complete system of forces acting on the body before writing down any equations.
9. Simplifying assumptions, such as that friction, stiffness of ropes, weights of certain bodies, etc., are being disregarded in any particular question, cannot be too explicitly stated.
10. Fancy names for technical terms are to be avoided.

C. Statics.

11. As the basis of the subject the parallelogram of forces should be assumed as an experimental result.
12. This should be *immediately followed by problems on the equilibrium of* bodies acted on by three concurrent forces, to be solved graphically.
13. The calculation methods should then follow the graphic methods, and should be applied to numerical cases in which four figure tables should be used, the angles 30°, 45°, 60 playing a very small part.
14. The Committee suggest that any of the following methods of dealing with the problem of parallel forces should be regarded as admissible:
    (α) By taking moments, assuming that a clear idea of the meaning of a moment and its mode of measurement have been obtained by experiment.
    (β) By the use of the funicular polygon, or by the equivalent geometrical device which deduces parallel from nonparallel forces.
    (γ) By defining a couple and proving from the parallelogram of forces that it is measured by its moment.
15. Both the graphical and analytical methods of determining the resultant of any number of coplanar forces should be taught, and numerical examples should be given.
16. It should be pointed out that all composition of forces assumes the existence of a rigid body to which the forces are applied, and tha failing the existence of such a body, composition of forces is unlawfu and indeed unmeaning.
17. The impression, that the weight of a body is in reality a single forc acting at its centre of gravity, should be guarded against. This an other cases where rigidity is assumed should be impressed on th beginner by contrasting rigid bodies with bodies which are not rigid.
18. *Machines.*—(α) The phrase "Mechanical Advantage" may preferabl be replaced by the term "Force-ratio," viz., the ratio of the resistanc to the effort when the effort is just overcoming the resistance.

(β) The consideration of work should be an essential part of the discussio of machines, and attention should at an early stage be given to "Velocity ratio" and "Efficiency."

(γ) Systems of pulleys should not be referred to by numbers.

19. When the equilibrium of two or more connected bodies or parts of a single body is considered, the principle of "Separate Equilibrium" should be distinctly enunciated.
20. It should be clearly pointed out that all the results of statics apply to cases of uniform motion.

D. Kinetics (commonly called Dynamics).*

21. *Velocity.*—The meaning of the phrase "velocity at a point" should be carefully brought out by means of the idea of "average velocity." Average velocity should be defined as "total distance / total time," and should not be assumed to be identical with the arithmetic mean of the initial and final velocities, or with the velocity at half-time, or with the velocity at half-way.
22. Much stress should be laid on Newton's first law and many examples given.
23. *Angular velocity* should receive attention, as, in connection with it, a great variety of interesting examples arise *e.g.* on the gearing of wheels.
24. *Acceleration.*—The velocity at any time should be represented graphically. This method should be used to illustrate the idea of acceleration, and the formulæ for uniformly accelerated motion should be obtained from the fact that the graph is in this case a straight line.
25. The formulæ for uniform acceleration having been proved as above, the fact that "the average velocity = the velocity at the middle instant" should be frequently employed in solving problems connected with such motion.
26. It should be expressly stated that all velocities with which we deal are necessarily relative to some base.
27. In explaining the parallelogram of velocities, it should be stated that we are proposing to determine the velocity of a point A relative to a base C from a knowledge of the velocity of A relative to a base B and of the velocity of B relative to the base C.
28. It should be permissible to treat elementary problems on the accelerations produced by forces by simple proportion

$$\left(\frac{\text{force acting}}{\text{acceleration produced}} = \frac{\text{weight}}{g}\right)$$

using the fact that a body's own weight produces acceleration g; and it should be allowable to postpone the consideration of mass until such problems have been discussed.
29. The poundal, when employed, should be used as an auxiliary unit, final results being reduced to gravitational units.
30. Atwood's machine should be regarded as a means of illustrating the laws of motion, and not as a practical method of finding g. The unsound method [mass moved $= m + m'$; moving force $= (m - m')g$] is to be condemned.
31. With the idea of preventing the notion that acceleration is always uniform, and having regard to the importance in physics of simple harmonic motion, it is advisable to consider such motion and the pendulum at an early stage, before the difficult parts of the work on projectiles and before oblique collisions.
32. Easy problems on the motion of a fly-wheel should form part of a course on elementary mechanics.
33. The phrase "centrifugal force" should be abandoned.
34. It should be clearly pointed out that the principle of the "Conservation of Energy" is not capable of universal application to mechanical problems,

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* The term "Dynamics" more properly connotes the whole science of force.

whereas the principle of the "Conservation of Momentum" can always be applied.

35. It should be pointed out that all the parallelogram laws are cases of a single law, that of the addition of vectors.
36. There should be no objection to illustrating the idea of a rate so as to lead up to the elementary ideas of the calculus.

E. Order of Teaching.

37. A short course of easy numerical trigonometry should precede the theoretical study of mechanics.
38. While several suggestions as to the order of (*a*) Statics, (*b*) Kinetics have been made above, the Committee does not wish to recommend that Statics should precede Kinetics, or *vice versa.*

REPORT OF THE M.A. COMMITTEE ON ADVANCED SCHOOL MATHEMATICS.

In considering this subject the Committee have taken account of the different classes of boys who study what may be termed "advanced mathematics" at school. These include candidates for army examinations, science students, engineering students, as well as boys who intend to read for mathematical schools at the universities. *Those who intend to take up engineering will* often be candidates for mathematical scholarships.

The Committee consider it desirable that, in order to secure the highest possible efficiency in teaching mathematics in schools, the schedules of examinations should be so framed that candidates may, so far as possible, all follow the same preliminary course.

The Committee think that a preliminary advanced course suitable for all classes of students should include :

(1) The following portions of Algebra : partial fractions, elementary manipulation with complex number, and geometrical applications thereof, the theory of equations so far as it treats of the numerical solution of equations, the notation and easy properties of determinants, the simpler tests of convergency, and the binomial, exponential and logarithmic series.

The following parts of Algebra should be excluded from the preliminary course :—The theory of numbers, probability, continued fractions (with the exception, perhaps, of simple continued fractions), the advanced theorems on inequalities, and on indeterminate equations, and summation of series.

The preliminary course should further include :

(2) An early introduction to the differential and integral calculus and a free use of the same in subsequent work. The use of differentials shall be permitted.

(3) Graphical illustrations of Demoivre's theorem. Simple work in trigonometrical series and factors.

(4) A treatment of the elementary parts of conic sections in which either the geometrical or the analytical method is used, that method being used in each particular case which is most suitable for the problem under discussion.

(5) The elementary geometry of the plane, cone, cylinder, sphere and regular solids, including practical solid geometry.

(6) An introduction to the dynamics of rotation (in two dimensions), viz., the motion of a rigid body round a fixed axis with uniformly accelerated angular velocity, together with other simple cases of the motions of rigid bodies.

The Sub-committee also wish to express their opinion of the importance of:

(7) A more intimate union between the teaching of Mathematics and

Science, whereby theoretical and practical work may be brought into relation with one another.

The above course would form an excellent preliminary course for scholarship candidates, who would then proceed to more advanced work in the various subjects so as to arrive at about the standard which is now required.

Regarding examinations for scholarships at the Universities in their twofold capacity, (1) as a means of selection, (2) as providing a guide to work at schools, there is no reason to doubt that the former function is efficiently performed by the papers as now set.

With regard to the second function, that of directing school work, if it s decided that the above preliminary course is on the right lines, it would be desirable for scholarship examiners to recognise it by:

(*a*) Setting questions involving applications of the integral calculus;

(*b*) requiring work in theoretical solid geometry as above indicated;

(*c*) abolishing the restriction to "methods of pure geometry," and leaving he method employed to the judgment of the candidate;

(*d*) discouraging the cramming of book work in advanced trigonometry by not requiring candidates to reproduce long and difficult proofs (such as hat needed to establish the infinite product form for the sine).

SCHOOL CERTIFICATE.

Board of Education,
Whitehall,
12th July 1904.

SIR,—I am directed to bring to your notice the enclosed copy of certain uggestions which have been submitted to the Board of Education by the Consultative Committee for a system of School Certificates. They have been ut forward by that Committee as the result of careful and prolonged onsideration after conference with representatives of the English Uniersities, of Associations of Teachers, and of Professional Bodies concerned with the question, and after consulting with persons specially conversant with the administration of similar examinations.

In view of the capital importance of these proposals in relation to econdary Schools, the Board of Education are anxious to obtain an xpression of the views of the Universities, and of other bodies interested, pon the important issues raised in them, and upon the solutions suggested y the Committee for consideration.

I am accordingly directed to request that you will be good enough to take uch steps as may seem to you best calculated to further the objects in iew, and to favour the Board not later than December 31st, 1904, or earlier hould it prove possible, with the outcome of such consideration as may be ccorded to the scheme.

In issuing the Report as it has been received from the Consultative Committee, it must be clearly understood that the Board refrain from the xpression of any view as to the desirability or feasibility of the proposals herein contained, and that they are not committed at this stage to any ction in the matter.

I am, Sir,
Your obedient servant,
ROBERT L. MORANT,
Secretary.

To the Secretaries
of the Mathematical Association.

BOARD OF EDUCATION.

CONSULTATIVE COMMITTEE.

Proposals for a System of School Certificates.

The Consultative Committee are of opinion :—

(1) That, with the object of diminishing the multiplicity of examinations affecting secondary schools, and of providing a test of adequate general education which may be widely accepted, a general system of school certificates is desirable.

The well-known term "Leaving Certificate" has been purposely avoided because it is to some extent misleading and is not unfrequently misunderstood.

(2) That it is not desirable that examinations for such certificates should be conducted by means of papers set for the whole country from a single central organisation.

This clause must be read together with clause (6). It will be found that it is not the intention of these proposals to ignore the influence of the State in the supervision of a general system of examinations. On the other hand it is important to prevent the evils which would almost certainly arise from the State having the sole responsibility in the matter. The desirability of bringing the examining body into closer relation with the teacher being recognised, it is obvious that in dealing with a population of more than 30,000,000 and a large number and great variety of schools, this object can only be effectively attained by the establishment of more than one examining body. The success of the Scottish and Welsh systems seems to be largely due to the limited number of schools with which they have to deal.

(3) That such examinations should be controlled by a recognised **examining body**, which should be either (1) a University, or (2) a combination of Universities, or (3) an Examination Board representative of a University or Universities and of the local authorities which are prepared to co-operate with them. It is desirable that whatever the examining body may be, teachers of schools should, where possible, be represented, and with regard to (3), that every such Board should contain a large academic element.

The proposal to form in some cases Boards representing local authorities and teachers in the schools as well as Universities may afford an opportunity for making an important new departure. Those local authorities especially which aid the schools and may perhaps pay the examination fees may be glad to be associated with a neighbouring University. There is further a growing body of public opinion in favour of associating the teachers in the schools with duties of this kind.

No general rule can be laid down requiring a school to be examined by a particular examining body. It may often be desirable that a school should be examined by the University or Board of the district in which it is situate. On the other hand a school may prefer to preserve or to create a connection with one of the Universities of Oxford, Cambridge or London. It is recognised that it would not be desirable, if it were possible, to disregard the non-local character of these Universities, or the position which their examinations occupy all over the country. The proposals are based on the assumption that it will ultimately be best for the secondary schools which are maintained or largely aided by local authorities to look to provincial examining bodies for the organisation of their examinations, and it is not improbable that local authorities may prefer their doing so; but in any case there will be a period of transition during which the new system and the existing University examinations will run side by side for all classes of schools, and the higher secondary schools will doubtless always retain complete liberty in the choice of their examining body.

(4) That recognition of these examining bodies should mean recognition by the Board of Education, acting on the advice of the Consultative Committee.

(5) That the following conditions should be required from schools which present candidates for school certificates :—

(*a*) Periodical Inspection. Whether this inspection be conducted by officers of the Board of Education, or by a University or other organisation recognised under Section 3 of the Board of Education Act, 1899, the report of the inspection should be communicated to the examining body.

(*b*) The communication of the course of studies pursued in the school to the examining body.

That an examining body should be at liberty to decline to examine a school if the result of the inspection has not been, in their opinion, satisfactory; or if the course of studies is such as they are not able to approve.

It is considered that in this connection inspection and examination should be treated as complementary one to the other. Inspection is required, in the first place to enable the examining body to judge whether the school is fitted to be admitted to the benefits of the system; but it is also required to enable the examiners to understand the aims and characters of the different schools, and so, on the one hand, to prevent the examination from becoming mechanical and rigid, and on the other to check any tendency in the school to direct its efforts too exclusively to success in the examination.

(6) That a **Central Board** should be established for England (excluding, for the present, Wales and Monmouth), consisting of representatives from the Board of Education and from the different examining bodies, whose duty should be to co-ordinate and control the standards of these examinations, to secure the interchangeability of certificates, and to consider and as far as possible to adjust the relations of the examining bodies and their spheres of external action.

Although absolute identity of standard between examinations conducted by different bodies and in different places may be an impossible ideal, practical equivalence can probably be secured. Further, more than one combination of subjects may be held to represent a good general education. It will be the duty of the Central Board to see that a sufficient minimum standard is maintained in each subject, so that certificates including these subjects, wherever given, may possess a generally recognised and interchangeable value, and further, that these certificates represent in each case a good general education.

(7) That the Board of Education should constitute this Central Board, as soon as, in their opinion, a sufficient number of recognised examining bodies have signified their willingness to be represented thereupon, and should take all steps that may be necessary to procure the acceptance of the certificates by the professional bodies.

(8) That since an examination held with the co-operation of the school in which a scholar has been taught is more likely to lead to a just estimate of the knowledge which he possesses than one held entirely by an outside body, the examination should be conducted in each school by **external and internal examiners**, representing respectively the examining body and the school staff.

(9) That the course of the work pursued by a scholar during his school career should be recorded and reported on by his teachers, and that this **school record and report** should be available for reference in deciding his fitness or unfitness to obtain a certificate.

The suggestion here is that an examiner, in any case in which he desires to do so, should be able to judge of the character of a candidate's school career. The school records and reports need not be of uniform pattern. What is required is that such materials shall be accessible as will enable an examiner to judge whether the scholar's school career has been satisfactory or not. These materials will include, at the least, the curriculum of all the classes which the candidate has attended, a note of the time he spent in each, and periodical reports of his industry, regularity, and progress.

(10) That the headmaster or headmistress of the school should certify that the candidate has received instruction during the necessary period, and is in his or her opinion, fit to enter for the examination.

(11) That the external examiner or examiners should have control of the examination, and should have a veto on the passing of any candidate.

(12) That the papers should be set by the external examiner, after consultation with the internal examiner.

> This consultation does not necessarily involve a series of personal interviews previous to the examination. Full information as to the courses of study pursued by the candidate would in the first instance be supplied to the examining body for the information of the external examiner. The books read, whether in English, classical or foreign literature, and the courses of history or geography studied, the practical work done, etc., would thus be reported. The internal examiner, also, would suggest series of questions, or indicate points upon which, in his opinion, questions, should be set; and in general the two examiners would correspond on the subject matter of the examination paper. The paper should, however, be finally made up by the external examiner on his own responsibility.

(13) That the allocation of work in reading and marking papers should be determined by the examining body, provided that papers which are near the minimum pass mark should be considered by both examiners.

(14) That oral and practical examinations should be conducted by the external and internal examiners acting in concert, who should, subject to section 11, jointly assess the marks for each candidate in this part of any examination.

(15) That in language examinations no special books should be prescribed, but that passages should be included from the books used in the school as well as unseen passages. That an oral examination should always be held in the case of modern languages.

(16) That there should be a **senior certificate** for pupils who have received not less than four years' instruction in a school or schools accepted for examination under section (5).

That there should be a **junior certificate** limited to pupils under 16 years of age who have received not less than three years' instruction in a school or schools accepted for examination under section (5).

> With reference to the number of years of instruction required, it will be desirable at first to give some latitude to the Central Board.

(17) That no certificates for honours or marks for special distinction should be given, but that it should be open to the examiners to recommend the award of scholarships within a school or group of schools, when called upon to do so.

(18) That scholars who are in a school which, in the opinion of the Board of Education, is unable to conform to these regulations, might be allowed to enter for the examinations under special regulations approved by Central Board.

UNIVOCAL CURVES AND ALGEBRAIC CURVES ON A QUADRIC SURFACE.

§ 1. The projective theory of conics is based chiefly on the fact that a conic is a unicursal curve, and the simplest analytical method of treatment is to take for reference a triangle formed by two tangents and their chord of contact, so that with suitable multiples of distances for coordinates the equation of the conic may be written in the form

$$y^2-xz=0.$$

The coordinates of any point on it may be expressed in terms of a parameter u by the equations

$$x/u^2=y/u=z,$$

and as u takes all values from $-\infty$ to $+\infty$ the point (x, y, z), or (u) as it will be called, describes the complete curve exactly once.

In order to form a mental picture of the curve we may put $z=1$ and nagine the lines $xy=0$ to be at right angles. The conic becomes a parabola, nd the parametric representation is the usual one. The point (0) is the ertex and (∞) is the infinite end of the axis, and the points (u) and ($-u$) re reflexions in the axis.

The parameters of two points (k_1) and (k_2) on the conic are the roots of the uadratic equation

$$u^2-(k_1+k_2)u+k_1k_2=0,$$

hich shows that the equation of the chord joining them is

$$x-(k_1+k_2)y+k_1k_2z=0.$$

Making $k_1=k_2=k$, we find that the tangent at (k) is

$$x-2ky+k^2z=0.$$

If (x, y, z) is regarded as a given point, this equation is a quadratic in k hose two roots u and v are the parameters of the points of contact of the angents from (x, y, z) to the conic and satisfy

$$x/uv=2y/(u+v)=z.$$

Hence u and v may be regarded as new coordinates of the point (x, y, z), nd the equation of any curve S becomes, after substitution for x, y, z, symmetric equation in u and v,

$$S(u, v)=0.$$

Conversely, every symmetric equation represents a curve, for it is expressible in terms of uv and $u+v$, and therefore in terms of x/z and y/z.

The new coordinates of any point are interchangeable, that is, the points u, v) and (v, u) are the same. So also every point on S has two pairs of oordinates, for if $S(u, v)=0$, then $S(v, u)=0$.

§ 2. $S(u, v)$ has no symmetric factors, for if it had, the curve S would be educible, and this is supposed not to be the case. But among all algebraic urves there is a special class for which $S(u, v)$ breaks into complementary actors $\phi(u, v)$, $\phi(v, u)$. Then it is sufficient to retain only one factor and rite the equation of the curve in the form

$$\phi(u, v)=0.$$

Of the two pairs of coordinates belonging to any point of S, only one atisfies this equation, and is therefore distinguished from the other pair hich satisfies $\phi(v, u)=0$. For this reason S is called a *univocal* curve* and s represented by a general irreducible unsymmetric equation in u and v.

The simplest univocal curve is given by

$$u-k=0,$$

epresenting the tangent at (k). The corresponding symmetric equation is

$$(u-k)(v-k)=0,$$

nd by means of the equations

$$x/uv=2y/(u+v)=z,$$

his becomes the previously found tangent

$$x-2ky+k^2z=0.$$

The univocal curve $u-v=0$

is the conic itself, and we have

$$z^2(u-v)^2=4(y^2-xz)=C_2 \text{ say}.$$

The general linear equation $au+bv+c=0$

ives the conic $(a-b)^2(y^2-xz)=\{(a+b)y+cz\}^2$

having double contact with $C_2=0$.

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* "Courbe univoque." See Humbert, *Journal de Math.* 1893, sér. 4, t. 9, p. 154.

§ 3. Let $\phi(u, v)$ be of degrees m in u and n in v, and suppose $m > n$; then we may write

$$z^m\{\phi(u, v)+\phi(v, u)\}=S_m(x, y, z),$$

and
$$z^{m-1}\{\phi(u, v)-\phi(v, u)\}=(u-v)S_{m-1}(x, y, z),$$

S_m and S_{m-1} being homogeneous polynomials of degrees indicated by their suffixes. On squaring and subtracting we get

$$4z^{2m}\phi(u, v)\phi(v, u)=S_m^2-C_2S_{m-1}^2,$$

showing that the univocal curve touches the conic $C_2=0$ at all common points, and further that *at all points of the curve, $\sqrt{C_2}$ has the value S_m/S_{m-1} which is rational in the point coordinates x, y, z.* It will be proved that this latter property is characteristic of univocal curves.

The points of contact with $C_2=0$ are given by the equation

$$\phi(u, u)=0$$

which has $m+n$ roots. Hence the curve $\phi=0$ is of order $m+n$, and $S_m^2-C_2S_{m-1}^2$ is divisible by z^{m-n}. The curve $S_m=0$ passes through the $m+n$ points of contact and $m-n$ times through the point (∞).

The univocal curve has a certain number of double points given by

$$\phi(u, v)=0, \quad \phi(v, u)=0, \quad u \neq v,$$

and these lie on both of the curves

$$S_m=0, \quad S_{m-1}=0.$$

§ 4. It is important to distinguish between curves which touch $C_2=0$ at all common points, and curves on which $\sqrt{C_2}$ is rational.

Let $S_{2n}=0$ be a curve of order $2n$ touching $C_2=0$ at $2n$ points. Through these a curve $S_n=0$ can be drawn, not meeting the conic elsewhere. Then $S_{2n}=0$ passes through all the intersections of $C_2=0$ and $S_n^2=0$, and accordingly,* there exists an identity of the form

$$S_{2n}=A_{2n-2}C_2+B_0S_n^2$$

which gives the general form of $S_{2n}=0$ satisfying the geometrical condition of $2n$-fold tangency. By similar reasoning, if $S_{2n+1}=0$ touches $C_2=0$ at the $2n$ points of intersection with $S_n=0$ and at the point of contact of the tangent $S_1=0$, then there exists an identity of the form

$$S_{2n+1}=A_{2n-1}C_2+S_1S_n^2.$$

In neither case has $\sqrt{C_2}$ a rational value on the curve, unless further conditions are satisfied.

Let $S=0$ be an irreducible curve at every point of which $\sqrt{C}$ has the rational value P/Q. Then the curve $P^2-Q^2C=0$ contains $S=0$ as one part; in other words, $S(x, y, z)$ is a factor of P^2-Q^2C. Substitute

$$x/uv=2y/(u+v)=z$$

and then we have, as the necessary and sufficient condition, that S must be a factor of $\{P+Q(u-v)\}\{P+Q(v-u)\}$, S, P, Q being now symmetric functions of u and v. Now if S were a factor of $P+Q(u-v)$, it would also be a factor of $P+Q(v-u)$ and hence a common factor of P and Q, contrary to the hypothesis that $\sqrt{C}$ has the definite value P/Q when $S=0$. Further S has no symmetric factors since the curve $S=0$ is irreducible. Hence S must be the product of two unsymmetric irreducible factors $\phi(u, v)$, $\phi(v, u)$, one of which divides $P+Q(u-v)$ and the other $P+Q(v-u)$.

Hence the univocal curves are the only curves on which $\sqrt{C}$ is rational, and the general equation of such a curve, multiplied possibly by a power of z, is

$$S_m^2-CS_{m-1}^2=0.$$

§ 5. The conditions for an n-ic curve $S(x, y, z)=0$, having n points of contact with $y^2=xz$, to be univocal, are that $S(2uv, u+v, 2)$ must be the product of two factors $\phi(u, v)$, $\phi(v, u)$. No conditions are required for $n=1$ or 2, for the general equation of a tangent line

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* Nöther's theorem, *Math. Annalen.* Bd. 6, p. 351, Bd. 52, p. 593.

$$x-2ky+k^2z=0$$

gives, on being multiplied by z,

$$y^2-xz=(y-kz)^2\,;$$

and again the general equation of a conic having double contact is

$$y^2-xz=(lx+my+nz)^2.$$

In each of these cases it is evident that $\sqrt{y^2-xz}$ has a rational value. In terms of u and v, the equation of the last curve is

$$luv+pu+qv+n=0$$

where $2p=m+1$, $2q=m-1$, and the univocal conic is the reciprocal with respect to $y^2=xz$ of the envelope of chords joining corresponding points of two projective rows on $y^2=xz$.

A single condition must be satisfied in order that a cubic curve having triple contact may be univocal. The general equation of a cubic curve touching $y^2=xz$ at $u=0$, 1, ∞ is

$$(ax+2by+cz)(y^2-xz)-y^2(x-2y+z),$$

but $\sqrt{y^2-xz}$ is not rational on it unless

$$b=\sqrt{ac}+\sqrt{a}+\sqrt{c}.$$

§ 6. The theory of univocal curves is intimately connected with the theory of algebraic curves on a quadric. Let the conic $y^2=xz$ be the section of the quadric by the plane $t=0$. Take the fourth corner of the tetrahedron of reference to be the pole of this plane, then the equation of the quadric may be written

$$t^2=y^2-xz.$$

It will be convenient to imagine the point $x=y=z=0$ at infinity in a direction perpendicular to the plane $t=0$, and to regard t as the distance of any point from this plane.

Any algebraic curve on the surface is partially given by its projection

$$f(x, y, z)=0$$

on the plane $t=0$. If f is a *general* polynomial in its arguments, this equation represents a cylinder cutting the quadric in an irreducible curve which is its own reflexion in the plane $t=0$; that is, the curve cuts each generator of the cylinder in two points equidistant from $t=0$. From the point of view of geometry on the quadric, this is a special property not possessed by the general curve; in fact, a general curve together with its reflexion in $t=0$ form the complete intersection with a cylinder, and the problem is to find under what conditions the intersection is reducible.

The equation of any surface which contains a curve K but not its reflexion must involve an odd power of t and hence can be reduced by means of the equation of the quadric to an equation of the form

$$P(x, y, z)-tQ(x, y, z)=0$$

representing a *monoid* surface.* This shows that the projection of K is a univocal curve, for at every point of it $\sqrt{y^2-xz}$ has the rational value P/Q. The reflexion of K lies on the monoid

$$P(x, y, z)+tQ(x, y, z)=0.$$

§ 7. The chief difficulty in the study of algebraical curves on a given surface is that more than one additional equation is required, for the general curve is not the complete intersection of any two surfaces, but must be defined as the part common to several. In the case of a quadric we are able, by means of univocal curves, to represent any curve on the surface by a single equation.

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* Cayley. *Coll. Papers*, vol. V., p. 7.

The parametric representation of the surface is given by

$$\frac{x}{uv}=\frac{2y}{u+v}=\frac{z}{1}=\frac{2t}{u-v},$$

so that the points (u, v) and (v, u), which in the plane were identical, are separated on the surface and are reflexions of each other in $t=0$.

By what has been shown, we have proved the fundamental theorem: *the equation of any curve on the surface is $\phi(u, v)=0$, where ϕ is a general polynomial in its arguments.*

The parametric curves $u=k$, $v=l$, are the generators

$$\left.\begin{array}{l} y+t=kz \\ y-t=k^{-1}x \end{array}\right\} \text{ and } \left.\begin{array}{l} y-t=lz \\ y+t=l^{-1}x \end{array}\right\}$$

respectively, intersecting in the point (k, l). If $\phi(u, v)$ is of degrees m in u and n in v, it represents a curve of order $m+n$ cutting each generator $u=$const. in n points and each generator $v=$const. in m points.

Suppose $m=n$; a typical term in $\phi(u, v)$ is $u^r v^s$, and if $r>s$ we have

$$z^m u^r v^s = z^{m-r}(y+t)^{r-s}x^s,$$

but if $r<s$ we have $$z^m u^r v^s = z^{m-s}(y-t)^{s-r}x^r.$$

In this way the equation $$z^m\phi(u, v)=0$$

becomes that of a surface of order m cutting the quadric in the $2m$-ic curve $\phi(u, v)=0$. Hence *every equation of the same degree in u and v represents a complete intersection.* From this follows Halphen's theorem* that any curve on a quadric together with a certain number of generators is a complete intersection: for if $m>n$ the function $\phi(u, v)(v-k_1)\,.\,.\,(v-k_{m-n})$ is of the same degree m in u and in v, showing that the curve $\phi=0$ and any $m-n$ generators of the system $v=$const. lie on an m-ic surface.

R. W. H. T. HUDSON.

MATHEMATICAL NOTES.

149. [A. 1. b.] *Note on some inequalities connected with the expressions*

$$w=\frac{\Sigma a\Sigma(a-b)(a-c)}{\Sigma a(a-b)(a-c)},\quad n=\frac{a+b+c}{a-b+c},$$

where a, b, c are three positive quantities in order of magnitude.

(i) n, w both lie between 1 and ∞.

(ii) n lies between w and 3; proved by showing that $n-w$ and $n-3$ are of opposite signs.

(iii) w lies between n and $\frac{1}{4}(n^2+3)$; proved by assuming n constant while w varies, or otherwise.

(iv) n lies between w and $(4w-3)^{\frac{1}{2}}$; proved from (iii), etc.

Hence the following series is in order of magnitude, the first term being ∞ or 1 according as the series is in descending or ascending order:

$$\infty \text{ or } 1,\ \tfrac{1}{4}(n^2+3),\ w,\ n,\ (4w-3)^{\frac{1}{2}},\ 3.$$

w and n are equal if either of them is equal to 1, 3, or ∞.

If $b+c=a'$, $c+a=b'$, $a+b=c'$, then a', b', c' are sides of a triangle; and conversely, if a', b', c' are sides of a triangle in order of magnitude we can put $a'=b+c$, $b'=c+a$, $c'=a+b$. Substituting for a, b, c in terms of a', b', c' in the above result, and making some slight changes, and finally dropping the dashes, we obtain the following:

If a, b, c are sides of a triangle in order of magnitude, and w, n stand for the same expressions as above, then the following series is in order of magnitude (commencing with ∞ or 2 according as it descends or ascends)

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* *Bull. de la Soc. math. de France*, vol. I., p. 19.

$$\infty \text{ or } 2,\ n+\frac{(n-2)(n-3)}{n-1},\ w,\ n,\ \frac{2(w+6)^{\frac{1}{2}}}{(w+6)^{\frac{1}{2}}-(w-2)^{\frac{1}{2}}},\ 3.$$

w and n both lie between 2 and ∞, and are equal if either of them is equal to 2, 3, or ∞.

F. S. Macaulay.

150. [K. 20. f.] *Napier's Rule of Circular Parts.*

Let ABC be a spherical triangle right-angled at C. Then another triangle having the same circular parts may be found as follows.

(1) Take a triangle symmetrically equal to ABC and place the two together with their sides AC in contact. They will form an isosceles triangle whose sides are $2a$, c, c, whose angles are $2A$, B, B and whose altitude is b.

(2) Take the polar of this triangle. Its sides are $\pi-2A$, $\pi-B$, $\pi-B$ and its angles $\pi-2a$, $\pi-c$, $\pi-c$, and its altitude is the supplement of the altitude of the previous triangle, *i.e.* $\pi-b$ (as can easily be shown).

(3) Take the colunar triangle of the last triangle. Its sides are $\pi-2A$, B, B, and its angles $\pi-2a$, c, c, and its altitude b.

(4) Split this triangle into two right-angled triangles. The angles of these will be c, $\frac{1}{2}\pi-a$, $\frac{1}{2}\pi$, and the sides opposite them b, $\frac{1}{2}\pi-A$, B.

If $A'B'C'$ denote this new triangle, we have therefore

$$a'=b,\ b'=\tfrac{1}{2}\pi-A,\ \tfrac{1}{2}\pi-A'=\tfrac{1}{2}\pi-c,\ \tfrac{1}{2}\pi-c'=\tfrac{1}{2}\pi-B \text{ and } \tfrac{1}{2}\pi-B'=a.$$

The triangle $A'B'C'$ has therefore the same circular parts as ABC, moved one place round.

In the penultimate step of this process we had an isosceles triangle formed of $A'B'C'$ and a symmetrically equal triangle with their sides a' in contact. By placing two such triangles with their sides b' in contact and repeating the processes, we obtain a third triangle having the same circular parts moved one place further round, and so on.

This method is not so elegant as the geometric one given in text-books on spherical trigonometry, but I would say that it is easier to remember, as I never find it possible to demonstrate the geometric method to a class without referring to the text-book.

G. H. Bryan.

151. [P. 3. b.] *Note on Successive Inversion.*

[The Apollonian circles of a triangle ABC are the three loci $b.PB=c.PC$, $c.PC=a.PA$, and $a.PA=b.PB$. They have for diameters the segments of the sides between the feet of the internal and external bisectors of opposite angles. They cointersect in two real points, inverse to the circum-circle, lying on the line joining the circum-centre and symmedian point. The pedal triangles of these points are equilateral.]

Inverting the property that the medians of an equilateral triangle cointersect at angles of 120°, it follows that the A-circles of a triangle are coaxal and cut each other at angles of 120°.

If any point P be taken on the sides of an equilateral triangle whose centroid is O, the concentric circle of radius OP will intersect the sides in five other points, symmetrically situated two on each side. No matter what their number or what their order, successive reflections of P with respect to the medians give these five points and no more.

Inverting; no matter what their number or what their order, successive inversions of a point P with respect to the A circles of the triangle ABC give five new points and no more.

The formal proof of the following property presents no difficulty and need only be stated; the pedal triangle of any point is similar to the pedal triangle of the inverse of that point with respect to either of the A-circles (M'Clelland, Chap. VI., Ex. 25, p. 148). It depends on the fact that the sides of the pedal triangles of P are proportional to $a.PA$, $b.PB$, and $c.PC$.

Thus there are generally eleven points that have their pedal triangles similar to the pedal triangle of a given point P. They are the five points obtained by successive inversions of P with respect to the A-circles; the inverse Q of P with respect to the circum-circle; and the five points obtained by similarly inverting Q. R. F. Davis.

[The above note arises from the discussion of a question suggested by the Rev. J. J. Milne; and assistance is gratefully acknowledged from Mr. C. E. Youngman and Mr. C. E. M'Vicker.]

152. [V. a.] *Echo from the Examination Room.*

"In the common system of logarithms the mantissa is the same for all numbers having the same magnificent digits."

REVIEWS.

Mechanics. By John Cox, Macdonald Professor of Physics in M'Gill University, Montreal; formerly Fellow of Trinity College, Cambridge. Cambridge University Press, 1904. Pp. xiv, 332. Price 9s.

Too often a school-book belongs to the class of books which are no books, but here excellent paper, large type, and beautiful diagrams and portraits give pleasure to the eye: nor does the author shrink from a well turned sentence or even an occasional joke.

Throughout the book may be seen the influence of Mach, to whom the work is dedicated, and to whom full acknowledgment is made in the preface. The author follows the historical development of the subject; indeed, as he states in his preface, the first book, "The Winning of the Principles," is to a great extent an abridgment of Mach's earlier chapters. In book II. the mathematical statement of the principles is taken up, followed in book III. by application to various problems, while book IV., on the elements of Rigid Dynamics, completes the work. This book, and indeed the whole work, is especially suited for the student of physics, to whom, even at an early stage in his career, a clear view of the outlines of dynamics of rotation is essential. It comprises a discussion of the compound and ballistic pendulum, flywheel, torsion balance, and Cavendish's experiment.

The author's attitude towards certain details of methods of instruction which have formed the subject of recent discussions here, may be described as, on the whole, conservative. Thus no special stress is laid on graphical methods. A tendency prevailed for a long period to ignore the just claims of graphical methods in elementary work, claims justified by this fact alone that, treated graphically, any one length or angle is as easily dealt with as another. Recently a tendency has been visible in certain quarters to lay undue stress on graphic methods almost to the exclusion of methods of calculation. Correspondence in *Nature* [May-June, 1904] has indicated divergence of opinion among experienced teachers as to the proper place of graphical work in elementary mechanics. It must certainly be acknowledged that many beginners are quite as ready to confuse the tension of a string with its length when attempting a graphical solution, as they ever are to mingle forces and moments when proceeding by the way of analysis.

Perhaps it would be worth while to enact that the "Frame diagram and Force diagram should always be entirely separate and should be drawn in different colours." No doubt, in dealing with an object such as "a weight of 10 pounds kept in position by a horizontal strut 3 feet long hinged to a wall, and by a string 5 feet long attached to a point in the wall," to draw a separate triangle of forces, is to many persons utterly revolting. However, well meant attempts to utilise the frame diagram, or part of it, as a force diagram are responsible for some queer blunders. Let the beginner have the pleasure and profit of inventing such labour-saving devices for himself. When he invents them he may safely use them

The more advanced graphical methods for dealing with coplanar non-concurrent forces certainly prove difficult to many students, partly because a certain power of viewing the question as a whole and deciding on a plan of action before putting pencil to paper is necessary. But after all, the process of writing down equations on no definite plan does not succeed in analysis so often as students sometimes appear to believe.

The author's attempt to clear up the difficulties inherent in the notion of 'mass' is not entirely successful. If an angel made the attempt his efforts would probably be criticised.

Professor Fitzgerald (*Scientific Papers*, edited by Professor Larmor, p. 452) observed "of all those dynamical quantities of which students are generally expected to form clear and distinct ideas without any actual experience of the things themselves, the most abstruse and the one about which the most metaphysical statements are made is 'quantity of matter' or 'mass.'"

One can avoid using the word at all (Professor Greenhill), or, with Professor Love, one may define the frame with reference to which accelerations are measured, and then *define the mass ratio of A and B as the reciprocal of the* ratio of the accelerations determined in A and B by their mutual action. There is nothing if one adopts the former course to prevent the observation being made that a book would, we cannot help believing, remain a book at the distance of the moon: or the conclusion being drawn that weight is not a characteristic property of a single body. But a middle course, in which we define mass and erect our structure of mechanics upon the definition in a manner which is at once unexceptionable and yet acceptable to the beginner, is one which I should be very glad to find, but which I do not think the author has achieved.

Another burning topic which for the most part the author leaves on one side is that of "practical" questions. He does not share the delusion that a question about a rod 12 inches long, supported at its ends, and carrying a weight of 5 ounces at its middle point, is converted into a practical one by simply expanding the rod into a beam 12 feet long, carrying 5 tons. The practical part of the matter would relate to the mode of supporting the ends, so as to ensure vertical reactions and an adequate amount of bearing surface.

Such details are outside the scope of the author's design, though judicious allusions to experiments and practice are frequent. A practical moral might, however, perhaps with advantage, have been pointed out in connection with the formula

$$T = T_0 e^{\mu\phi}$$

for the tension of a rope round a rough post. From a practical point of view, the feature of the formula is that it makes the ratio $T : T_0$ independent of the diameter of the post. This is because we have assumed the rope to be perfectly flexible. Real rope cannot be treated as perfectly flexible unless the diameter of the post is sufficiently large.

Most of us know the formula as one of which it is by no means easy to furnish an elementary proof, sound in every detail, and a student who found the proof given on page 200 unconvincing would have the author's sympathy (p. 201) and mine.

In his preface the author refers to his book as an experiment. It is one which, it may be hoped, has to-day a chance of obtaining the success which it deserves, a success which, not long ago, would have been impossible for a book which "*aims at no particular examinations.*" No book on elementary mechanics would appear better suited for use where the historical, mathematical, and physical sides of the subject are deemed equally worthy of cultivation. C. S. JACKSON.

The Tutorial Algebra. (Advanced Course.) By W. BRIGGS and G. H. BRYAN. Pp. viii, 612. 6s. 6d. 1903. (Tutorial Press.)

We are glad to see that this excellent volume is now in its second edition. There are no alterations in the present edition, but it will be found well up to date, and we may mention incidentally that it contains a chapter of nearly thirty pages on the graphical representation of functions and continuity of functions.

The Tutorial Statics. By W. Briggs and G. H. Bryan. Third edition. Pp. vii, 366. 3s. 6d. **The Tutorial Dynamics.** By the same Authors. Second edition. Pp. viii, 416. 3s. 6d. 1903. (Tutorial Press.)

These books seem to have made their mark, and are now re-issued with some alterations and additions. In the absence of any text-book on the market dealing with the elements of Mechanics from a practical standpoint, and as an introduction to what is a very difficult subject, these will serve as well as many of their rivals and better than most.

Examen des différentes Méthodes employées pour résoudre les Problèmes de Géometrie. By G. Lamé. Pp. 124. 1818. (Courcier.) Reprinted by A. Hermann, Paris.

Mr. A. Hermann is one of those philanthropists among publishers who do not object to the risk involved in issuing a reprint of a famous book, even when they may not see an immediate return for the outlay. He is eclectic in his choice, and the volumes he reprints have, it is needless to say, some other quality than that of mere rarity. A special interest attaches to the present volume in that although it was written at an early stage in the development of the author—when he was still an engineering student in the "Corps Royal des Mines"—yet scattered here and there throughout the book are to be found remarks which shew how strongly marked even at that age was the author's critical faculty. For this reason it is essentially a book which a teacher may read with pleasure and advantage. The price is five francs.

Mathematical Papers of the late George Green. Edited by (the late) N. M. Ferrers. Facsimile reprint. Pp. xii, 336. 20 frcs. 1903. (Hermann.)

George Green entered upon his short but brilliant University career at the age of forty. It was but natural that at such an age he would feel the course of training necessary for a high place in the Tripos to be exceedingly irksome. And this was the case, but his mathematical ability was so unmistakable that he came out fourth Wrangler, and two years later was elected to a Fellowship of his College. Unfortunately for the scientific world, after yet another two years he was touched by the hand of Death, the great leveller of all human aspirations. We can but feel a poignant regret when a life so full of promise is taken from among us, whether it be that of a man of mature age such as George Green or of younger types as Abel, who died at 26, or the even still younger Galois, who perished in a duel at 21.

The best known of his writings is an essay "On the Application of Mathematical Analysis to the Theories of Electricity and Magnetism," which was published by public subscription at Nottingham five years before he went to Cambridge. Here we find for the first time the word "potential" used to denote the function representing "the sum of all the particles" in the system divided by their respective distances from a given point.

Here, too, we find the famous formula in triple integrals which bears his name, the parent of analogous functions in magnetism and hydrodynamics. In applying the potential function to the discussion of the theories of magnetism and electricity the author proves many propositions that were later to be experimentally and independently demonstrated by Faraday. Nothing is more striking than the way in which theorems discovered by one writer are re-discovered years after by another. For instance the theorem on p. 64 referring to the density of the electric fluid on a certain conducting body was re-discovered twelve years later by three men to whom Green's work was unknown:—Chasles, Gauss, and W. Thomson. In the paper on the Reflexion and Refraction of Sound at the surface of separation of any two fluid media when a disturbance is propagated from one medium to the other, Green improves on the laborious methods of Poisson and reaches different results. When the velocity of transmisison of a wave in one medium is greater than in the other, by sufficiently increasing the angle of incidence in the first medium we can cause the refracted wave in the second to disappear. If we assume the vibration of the incident wave to follow a law such as that of the cycloidal pendulum, then, as the angle of incidence increases, the intensity of the reflected wave remains unaltered, while the phase of the vibration gradually changes. Duhem has pointed out a curious coincidence in connection with the equations of Green in this discussion.

Four geometers were pursuing their researches independently and simultaneously in France, England, Germany, and Ireland respectively, and each hailed "cette forme d'intégrale de l'équation des petits mouvements" as the key to the effects of total reflection in Acoustics and Optics. They were Cauchy, Green, Neumann, and MacCullagh.

Essai sur Hyperespace, Le Temps, La Matière, L'Energie. By M. BOUCHER. Pp. 204. 1903. 2 fr. 50. (Felix Alcan.)

This interesting little brochure deals with subjects on the borderland of mathematics and philosophy. The object of the author is to give as clearly as may be a popular account of what the words matter, energy, time and space should connote, but the greater part of the book is devoted to the consideration of the different forms of non-Euclidean space. The author is familiar with the current philosophical controversial literature on the subject, but we do not see from the list of authors referred to that he has embarked on any course of general reading such as would qualify him to engage the especial sympathy of mathematicians. He certainly quotes from M. Barbarin's little summary in the *Collection Scientia*, and from Poincaré's essay on non-Euclidean geometry, published in 1891. His acquaintance with English writings seems to be limited to Hinton's charming romances, to one of which he alludes as "a new era of thought" (p. 167) and more than once he draws the attention of the reader to the chapter on hyperspace in Mr. Rouse Ball's "Mathematical Recreations and Problems." An appendix of thirty pages gives an abstract of Mr. Stringham's paper in vol. iii of the "Américan (!) Journal of Mathematics. A scanty knowledge of English is no excuse for lack of familiarity with Mr. Russell's Foundations of Geometry, which has been translated into French, and as there is no mention of Klein, or Riemann, nor of Peano and Pieri, we may fairly conclude that the author has derived his knowledge from the philosophical articles and works in his list ranging from Plato to Lechalas, Renouvier, and Freycinet. M. Boucher would appear to be one of those to whom the idea of hyperspace is a fascinating topic, inasmuch as it gives us glimpses, as it were, of the possibility of other conditions of existence than those with which we are acquainted. The *motif* of his work is clearly the conviction that "la Raison se met d'accord avec cette appréhension presque unanime de l'humanite qui veut la distinguer de la matiere." From this point of view the book is well worth reading. As we have said his knowledge of current controversial literature is extensive, and he has compiled a very interesting account of the modern philosophical attitude towards the idea of non-Euclidean space. Some of our readers may care to send to the *Gazette* their opinion of the following "proof" by Galileo that we cannot have a series composed of an infinite number of terms, a proof which apparently receives the approval of M. Boucher. Among the first ten natural numbers there are only three squares; among the first hundred there are only ten; among the first thousand there are only thirty one; and so on. So that the number of squares is always less than the number of natural numbers. But an infinite series of numbers contains not only the square of each number, but the square of its square, etc. "La contradiction étant manifeste, il est évident que la suite des nombres ne peut être prolongée à l'infini."

The Teacher's Blackboard Arithmetic. Part I. By "TACT." Pp. 107. 1s. 6d. 1904. (Blackie.)

Although the scope of this little book is rather below that of the work done in most secondary schools, being limited to the first four rules, yet it is worth while calling attention to its merits as a carefully thought-out and systematic introduction to the ideas of number and the elementary operations. Some of the devices are quite ingenious and all are fully and clearly explained.

I Tre Problemi Classici degli Antichi. Problema Terzo:—Trisezione dell' Angolo. By PROF. BELLINO CARRARA. Pp. 60. 1904. (Fusi, Pavia.)

In this little pamphlet Prof. Carrara brings to a close his historical study of the three famous problems of antiquity. We note that although authors are quoted up to 1899, there is no mention of Klein's well-known "Vorträge uber ausgewählte Fragen der Elementargeometrie." In the trisection by means of the conchoid and circle Viviani is forgotten. We might enquire of our readers *en passant* if there is any construction for solving this problem by means of the *cissoid*, as is alleged in Chambers's *Encyclopædia* (cf. *Nature*, 1901, p. 400). For the benefit of our younger readers we venture to print the following note sent to Catalan by M.B. (Catalan, by the bye, referred to the fact that M.B. signed himself "*ancien Professeur de Mathématiques*" as a "*circonstance aggravante*").

To divide an angle AOB or any arc AB into three successive parts proportional to the numbers 3, 4, 5.

Take $OA = OB$. Complete the circle centre O, radius OA or OB, and draw the diameter AOC. Produce CB and OB to E and D respectively so that $BD = BE = OB$. Let EO cut the circle in F, DF cut the circle in G, and GE cut the circle in H. Then shall $BF : FH : HA = 3 : 4 : 5$.

An interesting section deals with various instruments for the practical solution of the problem, invented by Varignon, Fusinieri (spelled elsewhere Fusineri), Amadori, General Plebani, and—Good Arthur. With the name of the latter gentleman (apparently quoted in Enriques', (also spelled Euriques), "Questioni riguardanti la Geometria Elementare"), we admit, to our shame, we are not acquainted. Perhaps the name has been taken from an index, thus: "Good, Arthur." There are again many misprints in the names of authors and in the titles of books—a blot which marred, as we have previously pointed out, the value of the first two parts of this series. Taken as a whole, the three will be found full of interest and an adequate compilation dealing with the three classical problems which were the object of so much ingenious research and speculation in ancient times.

Einleitung in die Analytische Geometrie der höhern Algebraischen Kurven nach den Methoden von Jean Paul de Gua de Malves.

By P. Sauerbeck. Pp. vi, 166. 1902. (Teubner.)

De Gua's name is best known to the mathematician as the author of the "analytical triangle." He was Professor of Philosophy at a time when philosophy had a wider connotation than it at present enjoys, for under the influence of continental ideas the word science has gradually come to cover the ground of what once was called natural philosophy or simply philosophy. According to Mr. Rouse Ball, De Gua published in 1740 a book on analytical geometry, applying it to find the tangents, asymptotes, and singular points of an algebraical curve, without the aid of the calculus, and also shewing how singular points and isolated loops are affected by conical projection. The present essay has been inspired by Brill and Nöther's "Über die Entwicklung der Theorie der Algebraische Funktionen in alterer und neuerer Zeit." Dr. Sauerbeck, animated by a pious wish to give to the younger generation a faithful picture of the activity of a mathematician of a former time, has presented in modern guise the developments of De Gua's analytical methods, and has carried them on in the treatment of curves of the kind usually studied in modern text-books. A good idea and well carried out.

The Story of Arithmetic.

By Susan Cunnington, with a preface by Prof. W. H. H. Hudson. Pp. xiv, 239. 3s. 6d. 1904. (Swan, Sonnenschein.)

This useful little book will be found to contain a clearly written and succinct survey of what the authoress calls the alphabet of mathematical study. In addition to what we naturally would expect to find in such a volume, there are a few amusing and interesting chapters dealing with the mysticism and folk-lore of Arithmetic, the Arithmetic of Shakespeare, Arithmetic in modern literature, and so forth. To some of our readers the following extracts from Lewis Carroll's *Life and Letters* may be new. As requirements for the Mathematical School at Oxford he enumerates: "(A) A very large room for calculating Greatest Common Measure. To this a small one might be added for Least Common Multiple: this, however, might be dispensed with. (B) A piece of open ground for keeping roots and practising their extraction; it would be advisable to keep square roots by themselves as their corners are apt to damage others. (C) A room for reducing Fractions to their lowest terms. . . . (D) A large room, which might be darkened and fitted up with a magic lantern, for the purpose of exhibiting circulating decimals in the act of circulation. This might also contain cupboards fitted with glass doors for keeping the various Scales of

Notation." . . . So much for Arithmetic, but we cannot refrain from quoting one more passage: "(E) A narrow strip of ground railed off and carefully levelled, for investigating the properties of Asymptotes, and testing practically whether Parallel Lines meet or no: for this purpose it should reach, to use the expressive language of Euclid, 'ever so far.' This last process of 'continually producing the lines' may require centuries or more; but such a period, though long in the life of an individual, is as nothing in the life of the University." Miss Cunnington has done her work well, and this volume will not only give the reader much useful information, but a good deal of innocent enjoyment.

Experimental and Theoretical Course of Geometry. By A. T. WARREN. Pp. iv, 259. (Clarendon Press.)

Mr. Warren's excellent course has had more than a *succès d'estime*, for in a few months it has reached a second edition. The author has taken the opportunity of including all the propositions required according to recent changes of regulations for Pass examinations at the Universities. He has also introduced references to the corresponding articles in the Experimental course, so that the experimental course may be read *pari passu* with the present volume.

Constructive Geometry, being steps in the synthesis of ideas regarding the properties and relations of geometrical figures; arranged for a first year's course in science. By J. G. KERR, LL.D. Pp. 121. 1s. 6d. 1904. (Blackie.)

This is a well thought-out introduction to geometry, in which, according to the plan adopted in the teaching of to-day, geometrical notions are built up in the same way "as that followed in dealing with the elements of chemistry or physics." There is more effort than in most of the books we have seen to make the pupil reason about properties and to deduce fresh results. In the hands of a skilful teacher it should go far to produce the habits of enquiry and thought, without which real progress in the higher branches of the subject is more than doubtful.

New School Arithmetic. Part I. By C. PENDLEBURY, assisted by F. E. ROBINSON. Pp. viii, 206, xxi. 2s. 6d. 1904. (Bell.)

It is eighteen years since Mr. Pendlebury's Arithmetic was published and at once took a place among the most popular books in the market. The recommendations of the Committee of the Mathematical Association as to the teaching of the subject are the immediate cause of a revised issue of this text-book. Among the more striking features of this new edition is the prominence which is rightly given to the Metric system of Weights and Measures. The new method of multiplication and the Italian method of division are taught from the outset, as is the additive method in subtraction. The utility of rough approximations as an initial stage in the working of a problem is insisted on from the first. The decimalising of money is inculcated in the commercial sections of the book, and we are glad to see that the authors are not afraid of using an algebraic symbol when it is an advantage to do so. We need only say that Mr. Pendlebury has long secured his place, and we rather imagine he will keep it.

Longmans' Senior Arithmetic for Schools and Colleges. By T. F. G. DEXTER and A. H. GARLICK. Pp. ix, 554. 1904. (Longmans.)

We feel that this Arithmetic will be very popular in Training Colleges and similar institutions. Types of every imaginable kind of question are worked out, and but few difficulties are left for the student to ponder over, except in the sets of problems which follow the chapters or sections. The number of examples is appalling. There are plenty of questions on the theory of the subject, an example which might well be followed in other manuals. There is a chapter on logarithms, and the author also deals with elementary mensuration of the circle, cone, and pyramid. The following may be taken as an instance of the expositional power of the authors:—"*The detection of misplaced terms.* Students when working problems from an arithmetic, fail sometimes to get the right answer. Some then 'dodge' for the answer by 'trying it another way,' *i.e.* they change the order of their terms. There is no need to indulge in this reprehensible practice if the following method of detection is mastered—Take the right answer from the book and the wrong answer; form a ratio or fraction of the two answers, placing the incorrect one in the numerator and the correct one in the denominator; then the fraction thus formed will be (1) a square quantity if one ratio is misplaced; (2) the product of two square quantities if two ratios are misplaced." No explanation of this fact is vouchsafed. We might now understand why many teachers prefer their students not to have access to the answers. But the book is full of good points.

Preliminary Geometry. By R. Roberts. Pp. 56. 1s. 1904. (Blackie.)

This is likely to prove a useful little introduction to geometrical principles. It consists of a series of carefully graduated questions on the elementary parts of the subject, with test papers at intervals. The author claims that he has steered a middle course between discouraging the pupil by making the exercises too difficult, and, on the other hand, avoiding all need for mental effort. The claim is very largely justified.

Association of Teachers of Mathematics in the Middle States and Maryland. Bulletin No. I. July, 1904. Pp. 56. (Printed by the Association, New York.)

In the first year of what we hope will prove to be a vigorous existence of this new Society the number of members on its roll has nearly reached 300. The greater part of the Bulletin consists of papers read before the Association during the years 1903-4. They are on the "laboratory method" of teaching Mathematics; the Syllabus for Geometry in the grades; suggestions on topics for investigation by the Association; and on the "genuine applications" of Algebra. The first thing that strikes the reader is the closeness with which the improvements effected in our methods of teaching in this country have been followed by our brethren across the pond. The name of Perry is as well known to them as to us. Strained as the conditions of modern life may be in the States, we find in one of these papers the lament that "the school year is too short in most of our widely patronized private and endowed schools by reason of the vacation plans of their well-to-do patrons, and the luxurious tendencies of the times, which are inimical to strenuous application." It is a pity that the writers of some of these papers do not pay a little more attention to the virtues of literary form. Occasionally we find an outburst of feeble flamboyancy, but as a rule the style is bald and tame. Behold this purple patch! "It is indeed a source of satisfaction to know that so many individuals have awakened from a comatose state of dull perceptions sufficiently to realise that algebra has applications in physical science! We hear the cry 'Eureka' from the shores of Lake Michigan. This is the fountain of youth to the teacher, a promised land to the children after generations have died in the wilderness of old-fashioned mathematical teaching." But apart from the form of the papers there is little to be said against their content. Enthusiasm inspires them, and it is an enthusiasm which is clearly being organised and controlled. It is interesting to note the nature of the problems to which the attention of the American teacher is directed in the paper on topics for investigation by the Association. "What should be the aim of mathematical teaching? How can the bookwork in algebra and geometry be reduced to a minimum? How much exercise work (*examples*) should be done? How much time is necessary for teaching algebra and geometry intelligently (!) to the average class? Is it necessary to give so much time to Arithmetic, and do the results of arithmetical teaching justify the present large expenditure of time? What should be the character of the examinations? What is the maximum number of students in a class which a teacher of mathematics can instruct properly? What should be the qualifications of a teacher of mathematics? Which are the most essential chapters of algebra, geometry, and trigonometry? What parts of book-study could be entirely omitted? Which are the best methods for leading students into original mathematical work? What kind of original work is most suitable for the average secondary student? Which chapters of secondary school mathematics have practical value? Will the introduction of the 'laboratory method' reform mathematical teaching?"

We wish to the new Association a long life and all the success that the importance of its aims demands.

Aufgaben aus der Niederen Geometrie. By I. Alexandroff. Pp. v, 123. 1904. (Teubner.)

This is the German edition of a useful introduction to methods of attacking the solution of geometrical riders. It was translated from the Russian into French some years ago, and was at the time reviewed in these columns. We are glad to call attention to the German edition, as to the student who knows French and wishes to pick up German mathematical terminology, the two volumes may prove of considerable value, quite apart from the intrinsic merits of the book.

Lehrbuch der Differenzenrechnung. By D. Seliwanoff. Pp. vi, 92. 1904. (Teubner.)

This is a handy little introduction to the Calculus of Finite Differences, simply and clearly written. The first part deals with introductory matter, interpolation, and the approximate values of certain integrals. The second treats of summation, the J. Bernouilli Function, and Euler's summation formula with its applications. The third is devoted to linear difference equations of the first order, and linear difference equations with constant coefficients—the whole being subject matter of Boole's first eleven chapters, but treated lightly and on a smaller scale.

Lehrbuch der Analytischen Geometrie. By O. Fort and O. Schlomilch. Vol. I. **Analytische Geometrie der Ebene.** 7th edition. Revised by R. Heger. 1904. (Teubner.)

This, the seventh edition of a book, the special characteristic of which is its clearness and simplicity of exposition, is revised and improved in various details by Professor Heger of the Polytechnic at Dresden. The second, third, and fourth editions were edited by O. Fort, and it is one-and-twenty years since the fifth edition was revised by Professor Heger. The ninth and tenth chapters deal with lines of a higher order than the second, with transcendental lines in general, and spirals and the usual roulettes in particular.

Méthodes de Résolution et de Discussion des Problèmes de Géométrie. By G. Lemaire. Pp. 224. 1904. (Vuibert et Nony.)

This is a good representative of a type of compilation of which we have but few instances in this country. It covers much the same ground as Petersen and Alexandroff, both of which are available in French. There is something to be said for the view that a boy taught, as until recently he has here been taught, will not necessarily take "in his stride" and without preparation in some detail, such a grasp of the different methods by which a problem may be attacked as he will after the study of such a volume as this. The names of the sections will sufficiently indicate the lines on which the book is constructed: Loci; method of the intersection of loci; determination of a line (envelopes, lines through a fixed point, lines parallel to a given direction): translation, rotation, symmetry, method of similar figures; figures similarly and similarly situated, inversion; transformation and division of figures. Many of the solutions of problems are accompanied by discussion in detail, most unusual with us, but a process especially commendable in the case of intelligent boys, who often find this method a source of much inspiration.

Advanced Course in Algebra. By W. Wells. Pp. viii, 581. 6s. 6d. 1904. (Heath.)

The course that is covered in this volume is hardly what we should call "advanced" algebra in this country, and yet we are told it is intended to meet the needs of "Colleges and Scientific Schools of the highest rank." For the book contains no more than is to be found in Smith's *Treatise*, or Hall & Knight's *Higher Algebra*—not so much, indeed, for the subject is not treated with the same detail. It takes nearly four hundred pages to bring the student to the end of the chapter on the binomial theorem, and only one hundred and fifty pages are devoted to permutations and combinations, probability, continued fractions, summation of series, theory of numbers, determinants, and theory of equations. Simple, simultaneous, and quadratic equations take up one fifth of the book. Graphs are well to the fore in dealing with equations generally, the theory of the irrational number, derivatives, multiple roots, Sturm's theorem, and discontinuous functions. The chapter on involution contains the binomial theorem for positive integral exponents, proved by induction. The explanations are generally thoughtfully and clearly put, but now and then are overdone—as, for instance, in the case of the finding of the square and cube roots of an algebraical expression. We notice as an example of the author's reluctance to enter into detail, that he is content with instructing the reader to reject the negative value of n, where n is the number of terms in an A.P., without attempting to give any interpretation of the apparent anomaly. The following is the only definition of a function we can find: "A function of a number is any expression which contains the number. Thus the expression $2x^2-3ax+5a^2$ is a function of x." If it were worth defining at all it was worth doing better than this. But on the whole the book is sound, carefully considered, and equal to, if not better than, any American text-book we have seen. The questions at the ends of the chapters seem well selected, and the book itself is well printed and nicely bound.

The Collected Mathematical Papers of James Joseph Sylvester. Vol. I. (1837-1853). Pp. 650. 18s. net. (Cambridge Univ. Press.)

"As an artist delights in recalling the particular time and atmospheric effects under which he has composed a favourite sketch, so I hope to be excused putting on record that it was in listening to one of the magnificent choruses in the 'Israel in Egypt' that, unsought and unsolicited, like a ray of light, silently stole into my mind the idea (simple, but previously unperceived) of the equivalence of the Sturmian residues to the denominator series formed by the reverse convergents. The idea was just what was wanting, the key-note to the due and perfect evolution of the theory." "It is, as I take pleasure in repeating, to a hint from Mr. Cayley, who habitually discourses pearls and rubies, that I am indebted for the precious and pregnant observation. . . . A distinguished mathematical friend in Paris communicated to me with great admiration Professor Hesse's result overnight. I ventured to affirm that to one conversant with the calculus of forms, the problem could offer no manner of difficulty. An hour's quiet reflection in bed the following morning or morning after sufficed to disclose to me the true principle of the solution. . . . Guided by an instinctive sense of the beautiful and fitting, in a happy moment I have succeeded in grasping this much-wished-for representation, with which I propose now and forever to take my farewell of this long and deeply cogitated theorem. A morning or two after the enquiry reached me, in a walk before breakfast by the side of the ornamental water in St. James's Park (a time and place by no means, according to my experience, unfavourable to the aspirations of the analytic muse) I had the satisfaction of falling on the rather piquant demonstration above given. . . ."

The above, taken at random, extracts from this stately tribute to the genius of Sylvester, the first volume of which lies before us, effectively illustrate certain characteristics of one whose name is written in letters of gold on the roll of English mathematicians. And although they are taken from papers which were written before he had reached the plenitude of his powers, the same characteristics stand out in deeper relief as time goes on, and will be as dominant in the last of the volumes of his collected papers as they are in the first. They exhibit in the most striking manner, occurring as they do in papers dealing with the most abstruse forms of mathematical thought, Sylvester's intense enthusiasm for his subject, and the fine spirit of loyalty and admiration for the intellectual powers of his great collaborateur in the development of the theory of invariants by which the writer was ever animated. It is rare indeed to come across notes so entirely, so innocently personal in the pages of a mathematical memoir.

The present volume contains 68 papers, varying in length from one or two to 160 pages. The longest is entitled "On the Theory of the Syzygetic Relations of two Rational Integral Functions, comprising an application to the Theory of Sturm's Functions. . . ." It also contains his dialytic method of elimination, the Essay on Canonical Forms, and some of the earlier of the investigations in the theory of Invariants. It is most interesting to trace the steps in the development of the latter theory, in which Sylvester played so important a part. "The theory of Invariants," says MacMahon, "sprang into existence under the strong hand of Cayley, but that it emerged finally a complete work of art, for the admiration of future generations of mathematicians, was largely owing to the flashes of inspiration with which Sylvester's intellect illuminated it." For thirty years it was the main object of mathematical research both here and on the continent, a fact to which the names of Cayley, Sylvester, Hermite, Aronhold, Clebsch, and Brioschi bear sufficient testimony.

Our younger readers may wish to know more about this remarkable personality, and to them the following references may be useful, although they do not pretend to be complete: *The Eagle*, vol. xix, pp. 596-605 (with portrait); *Proc. London Math. Soc.*, vol. xxviii., pp. 581-586 (written by J. J. Walker); *Bull. of the American Math. Soc.*, 1897, pp. 299-309 (written by Fabian Franklin, an old pupil at the Johns Hopkins University); Nature Series of Scientific Worthies, *Nature*, vol. xxxix. (written by Cayley, with portrait); obituary notice in *Nature*, 1897 (written by Major MacMahon). *Obituary notices also appeared in the American Journal of Mathematics*, which he was instrumental in founding, and in *Science*, the latter being from the pen of one of his pupils, Prof. Halsted. There is a short biographical notice in the new edition of the *Encyclopaedia Britannica*, quite unworthy of so brilliant a mathematician, and ending with a needlessly ungenerous sneer at Sylvester's poetical capacity.

SOLUTIONS.

497. [R. 4. c.] *Two uniform rods AB, AC of weights W, W' are rigidly joined at A, and AB is placed on a rough peg: shew that there is, in general, a length $\ngtr \frac{\mu W'p}{W+W'}$ on AB, such that if the peg touch AB anywhere within this space the system will be in equilibrium, p being the length of the perpendicular from C on AB and μ the coefficient of friction.* C.

Solution by R. F. Muirhead.

Let B' and C' be the mid-points of AB, AC, and G the centre of gravity of the system, so that G lies in $B'C'$ and $B'G : GC' = W' : W$.

Let GM and $C'N'$ be perpendiculars let fall on AB, so that $C'N' = p/2$.

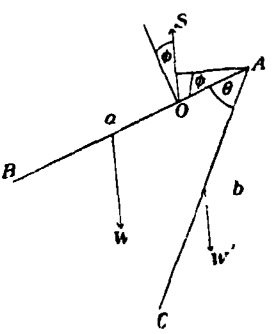
We have $$GM : C'N' = B'G : B'C' = W' : W' + W;$$

$$\therefore\ GM = \frac{p}{2} \times \frac{W'}{W' + W}.$$

For equilibrium, the vertical through G must pass through the peg, and give the direction of the reaction. For limiting equilibrium, this direction must make an angle with GM equal to the angle of friction.

Hence the two limiting positions of the peg cut off from AB a length $= 2\mu \times GM = \mu p W' (W' + W)$.

Solution by A. W. Poole.



(1) Let rods be just about to slip down. Also $\tan\phi = \mu$. $\theta = \angle BAC$. Action at peg must be vertical, for W and W' are vertical.

We have $$S = W + W',$$

and taking moments about A

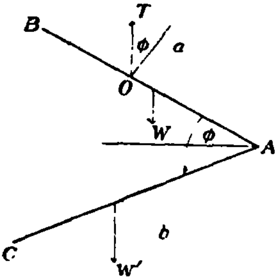
$$Sx\cos\phi = W\frac{a}{2}\cos\phi + W'\frac{b}{2}\cos(\theta+\phi),$$

where x is distance of peg from A;

$$x = \frac{Wa\cos\phi + W'b\cos(\theta+\phi)}{2(W+W')\cos\phi}$$

$$= \frac{Wa + W'b[\cos\theta - \mu\sin\theta]}{2(W+W')}$$

(2) Let friction be limiting as before, the rods slipping in the other direction.



$$T = W + W',$$

and taking moments about A

$$Tx' \cos\phi = W\frac{a}{2}\cos\phi + W'\frac{b}{2}\cos(\theta - \phi);$$

$$\therefore \quad x' = \frac{Wa + W'b[\cos\theta + \mu\sin\theta]}{2(W + W')},$$

where x' is distance of peg from A;

$$\therefore \quad x' - x = \frac{2W'b\mu\sin\theta}{2(W + W')}$$

$$= \frac{W'\mu p}{W + W'};$$

$\therefore$ for equilibrium the peg must be in this length.

ERRATUM.

P. 3 ; line 13. For 2 read *c*.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) For Sale.

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I.-IX. Edited and Published by E. Hendricks, M.A., Des Moines, Iowa, U.S.A.

[With Vols V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by Artemas Martin, M.A. (Erie, Pa.)]

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. Runkle, A.M.

Proceedings of the London Mathematical Society. First series, complete. Vols. 1-35. Bound in 27 vols. Half calf. £25.

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Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

BOOKS, ETC., RECEIVED.

The American Journal of Mathematics. Edited by F. Morley. Vol. XXVI., No. 2. April, 1904. $1.50. (Kegan, Paul.)

[On Nullsystems in Space of Five Dimensions. *J. Eiesland.* On the Forms of Unicursal Quintic Curves. *P. Field.* Determination of Algebraic Curves whose Polar Conics are Parabolae. *E. Kasner.* On certain Conics connected with Trinodal Quartics. *A. B. Bassett.* A Geometric Proposition. *E. Lasker.* Congruences of Tangents to a Surface and Derived Congruences. *L. P. Eisenhart.*]

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THE

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EDITED BY

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Vol. III. October, 1904. No. 47.

OBITUARY.

R. W. H. T. HUDSON.

It is with the deepest sorrow that we record the tragic death of Ronald Hudson, at the early age of 28, while spending a holiday with a friend in Wales. He and his friend started from the Penygwryd Hotel early on Tuesday morning, September 20, and after climbing one of the Glydyrs descended to the foot of the Devil's Kitchen by Lake Idwal, which they reached about one o'clock. Ronald Hudson was an enthusiastic rock climber, and would not be dissuaded from trying part of the ascent, although his companion was not sufficiently expert to go with him. A short half hour later he apparently dislodged a large rock, fell a considerable distance, and in all probability was instantly killed. When a search party with much difficulty found him some time after, he was dead. On the following Saturday he was laid to rest in Wandsworth Cemetery, by the side of his mother, whom he had lost when he was six years old. The heartfelt sympathy of his large circle of friends goes out to the sorrowing family, to his father, aunt and sisters, who have followed along the same path of knowledge as their brother with no less determination and success.

It is impossible not to associate this terribly sad accident with that which befell his intimate friend A. P. Thompson little more than a year ago. They were educated in the same school, but were never in the same class; and at Cambridge they were drawn together not so much by their common devotion to mathematics as by their love of music. On the evening of the day that proved fatal to Thompson they had arranged to go over some new music together. Hudson often repeated that he owed much to his friendship with Thompson, and there is no doubt that each exerted a profound influence on the character of the other.

Ronald William Henry Turnbull Hudson was named after his father, Professor W. H. H. Hudson, and his grandfather, Robert Turnbull; but his which propositions, formerly treated as universal and almost axiomatic, have been shown to depend upon subtle and complex conditions. This has unquestionably created a good deal of embarrassment for teachers of elementary mathematics. It is impossible to impose rigour on the beginner, and most undesirable to attempt it. No less an authority than Professor Klein can be cited in support of this view, which, in fact, constitutes the principle underlying the recent reforms in the teaching of geometry.

Even at the Ecole Polytechnique, where the students can hardly be called beginners, the attempt which was made to treat the Calculus from the outset on lines of strict rigour, had, I am told, to be abandoned. No doubt in commencing the Calculus one may now and again draw attention to some mathematical curiosity, such as that which Professor Pascal adopts as his first example :

$$\underset{t=0}{L}\ \underset{n=\infty}{L}\ \frac{\sin^2(x\,.\,n'\,.\,\pi)}{t^2+\sin^2(x\,.\,n'\,.\,\pi)},$$

which is zero when x is commensurable and unity when x is incommensurable. But if we are to put aside refinements which a beginner cannot appreciate, we shall find it very difficult to avoid enunciating theorems as generally true without regard to the qualifications under which alone they are true. In our difficulty we may turn for help to history. The evolution of the individual is a recapitulation of the evolution of the race. Now a science begins with facts and special methods, and ends with critical exactness and with generalised methods. As teachers of pre-university work, it is our main business to familiarise our pupils with facts and the mode of dealing with facts.

To my mind, the due discrimination on broad lines between the work proper to a school and the work proper to a university is a problem which calls for our most serious attention, and the solution of which will dispose of many of our difficulties. As teachers of school work, we must frequently be content with the certainty that a method is available in a particular case. Thus I should advocate the use of the method of differentiation in obtaining particular expansions, whose convergency may be examined in each individual case rather than purporting to furnish any general method of expanding functions. We shall have to lay increased stress on intuition. With that tendency to fetishism which most of us inherit, we have been sometimes apt to unconsciously attribute a sort of objective value to certain "pieces of book-work." To reproduce Duchayla's proof was to perform a meritorious action.

Under the influence of Mach we have come to doubt the objective value of this or any similar performance, and to consider rather whether the "proof" has really helped the student to realise more vividly, distinctly, and precisely that the proposition is true. Has it aided him to form a useful 'concept'?

It must, of course, be acknowledged that the truth and beauty of a demonstration may flash upon the mind after, but not until after, long study. But these matters are not for the beginner, whose mathematical courage must not be damped by premature encounters with difficulties which he cannot surmount. Further, one of the idlest fears that can beset a teacher is the notion that he can possibly make his demonstrations too easy or his explanations too clear. Now, it must, I think, be admitted that in many cases models and experiments may be employed with advantage in aiding a pupil to grasp a demonstration and in developing mathematical insight. Very often, as soon as a clear mind picture of the problem has been formed, its solution is obvious. The faculty of visualising a somewhat complicated object without the aid of models can be greatly developed by the judicious use of models, and it shares with mental arithmetic the distinction of being a most valuable mental power, and of being almost entirely uncared for in our educational system. Dr. Todhunter is said to have observed that no one who required a model or experiment to enable him to grasp a geometrical or mechanical theorem could ever be a mathematician. I believe that this is perfectly true. But how many of us who are lovers of mathematics would venture to call ourselves mathematicians? If one is not capable of being a mathematician in the high sense of the word, is one therefore to be prohibited from getting at the truth of things as best one can? I may perhaps give a personal reminiscence, which compromises no one but myself. When I went in for the mathematical tripos, I had no real conception of the action of a convex lens. I could have doubtless worked out various questions. But if you had said, "A candle is placed in front of a convex lens, and a real inverted image formed on a screen. The eye is then placed at the point where the image was formed. What will the eye see?" I could not have told you.

But, I may be told, you are asking that proposals for change should be welcomed without telling us what the proposals are. I am not inclined to go very far in the way of specific suggestion at present The mode of teaching chemistry and physics, and, in particular, the proper extent and aim of a school course in physics, is still far from being settled. I think teachers of mathematics may play a very useful part in the discussion, particularly in encouraging simplicity of method. Apart, however, from the general statement that teaching of geometry, trigonometry, and, in particular, mechanics, has, I submit, much to gain by becoming more concrete, I will venture on one specific suggestion, namely, that for young boys the study of Heat should be replaced by the study of Optics, and that the experimental work in Optics should be under the control of the mathematical staff. The laws of reflection and refraction furnish us with admirable and interesting exercises in practical geometry; and this work should give definiteness of ideas which will help the subsequent study of aberration, if that is reached at the university stage. Lenses are cheap; rough, but working, models of a sextant, a low-power telescope, an optical level or square and the like, are easily made. I produce a rough model of a sextant, not as being worthy of exhibition—for, if put forward with that intent, it would justly arouse ridicule—but in support of my contention that, with the roughest materials and the simplest tools, a boy of 12 could construct in half an hour a working model capable of measuring angles within a degree, and that a boy of more than 12 could learn a good deal from it. The problem how to make tenth of an inch section paper furnish a scale of degrees gives a little exercise in trigonometry. I attach importance to making rough models with one's own hands. Centering error, index error, error due to the mirrors not being at right angles to the plane of the arc, are very real when, for example, the latter error throws the image right off the second mirror.

I learn from Mr. Fletcher that he has put these views as to the teaching of optics into practice at Liverpool, for some time past, with complete success. If the treaty of alliance Mr. Thwaites proposes is entered into, the teaching of physics will become more mathematical; and here, at any rate, we believe that this will be well. On the other hand, teachers of mathematics will have to lay more stress on the art of stating problems, which is at least as important as the art of solving them. It has been said that hardly any person would fail to solve a strategical problem correctly if it were stated to him correctly, and divested of all irrelevant details. But the task of the commander is, as I suppose, out of the maze of facts, rumours, reports, surmises, and possibilities, to divine what the real problem is; and this is the task of the investigator in most subjects. Mrs. Boole has spoken in a delightful book of the importance of drawing attention, during the early stages of education, to the question whether a particular fact is or is not relevant for a particular purpose.

I should like to give a couple of concrete instances, one on each side, of the kind of misapprehension which a judicious admixture of theoretical and practical work would tend to prevent. The first I give simply as a legend, not to say a libel. When buffers were first proposed to take up the recoil of a gun on firing they were objected to; and it is said that their adoption was delayed on the ground that the explosion of the charge communicated a certain definite momentum to the gun, and that the whole of this momentum must be equally communicated, so to speak, to the ground, irrespective of any details of internal construction of the gun carriage. Hence, buffers are useless. Here was a conclusion based upon premises which were true enough. We, however, see at once that the *rate* of destruction of momentum is the decisive feature of the problem, and this is just what the buffer alters. If the objector would have jumped from a height of thirty feet, first into a net and then on to a stone pavement, losing the same amount of momentum in each case, he might, either in this world or the next, have reconsidered his argument.

On the other hand, in remonstrating with a student for producing some entirely new views on specific gravity, I once said, "But you must have often done this experimentally at school." The answer was: "Yes, sir: but we always had the formula given us, and only had to think about the weights." I do not much, if at all, object to his having the formula given him, but I do object to his only thinking about the weights. We must not forget that, while experiment is good, the discoverer thinks *before* he experiments as well as after. Here, again, I think teachers of mathematics may exert a beneficial influence. Some people fear that that the fusion of physics and mathematics in public school teaching may be like the fusion of the tiger and the lamb. But I hope I have shown that the high contracting parties will meet on a footing of equality, and should both benefit equally. An apprehension is sometimes expressed as to the cost of laboratories and appliances for teaching physics. I am happy to think that many people whose opinion on a question of science is of real weight, agree with me in thinking that we have had far too much in the past of the "instrument on polished mahogany base, with all brass parts richly lacquered," for demonstrating that a weight hung up by a string will oscillate to and fro if disturbed from its position of equilibrium. Home-made apparatus—cardboard, seccotine and scissors, sixpenny lenses in cardboard tubes, paper scales, levers and pulleys made in the school workshops are what we want. The fact is often lost sight of how rough, comparatively speaking, were some of the methods and results of the great pioneers.

Mankind may be divided roughly into two classes: those who understand their job and those who do not, generally because they do not know what understanding a subject means. Our first object as teachers is to teach people what understanding means, in order that in after life they may try to belong to the former class, be their job what it may.

I have attempted to put before you the arguments which have convinced me that a judicious admixture of mathematics and physics will aid us more powerfully than stronger separate doses of each to effect our object.

C. S. Jackson.

ON THE AXIOMS AND POSTULATES EMPLOYED IN THE ELEMENTARY PLANE CONSTRUCTIONS.

By the elementary plane constructions we mean the bisection of angles and segments of lines and the drawing of perpendiculars. The constructions given below are extremely simple, and it is very possible that some or all of them have been given before; but they seem to be of sufficient interest to be reproduced, for the reason that, if they are not new, they are not generally known.

The axioms employed are those tacitly assumed or explicitly stated in Euclid in relation to straight lines and angles, the axiom of parallels and the axiom that "two straight lines cannot enclose a space" being excepted. This will probably be accepted as a sufficient explanation, since we are not at present concerned with any abstruse analysis of axioms.

For the axiom of superposition we may substitute the axiom that the plane is uniform, possessing identical properties throughout, and that its uniformness about a point is independent of phase or sense of rotation. From this axiom many propositions follow at once, such as the equality of vertically opposite angles, the equality of the base angles of an isosceles triangle with its converse, and the identical equality of triangles having certain parts equal. Other propositions can be proved, such as the identical equality of triangles with all three sides given, or that of right-angled triangles having given the hypotenuse and one side, or a side and the opposite angle. It also follows from the axiom that all or none of the straight lines of the plane are closed. If they are not closed two straight lines through a point A do not cut again. If they are closed two straight lines through A may or may not cut again in another point B. If they do cut again in B, then every straight line through A passes through B; the lines AB are all of equal length; and they are all perpendicular to and bisected by one and the same straight line.

The postulates employed are that a straight edge may be used for drawing the straight line through any two given points, and for prolonging a given segment of a line to any desired extent; and that the dividers may be used for transferring segments, that is, for marking off from any point of a straight line a length equal to that of a given segment.

The postulate that a circle may be described with any centre and any radius is not employed, so that no assumption is made about the intersections of circles.

The proofs of (2), (3), (4), (6) are evident, and are consequently omitted.

(1) *To bisect a given angle.* (Fig. 1.)

Let A be the vertex. Choose two points B, C in one arm, and from the other arm cut off $AD=AB$, and $AE=AC$. Join BE, CD, cutting in X. Join AX; then AX bisects the angle A.

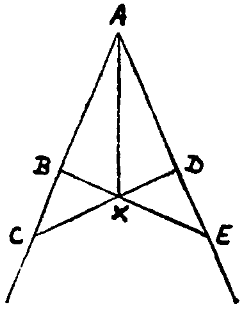

Fig. 1.

If straight lines have at most one point of intersection this construction is unique. If they have two points of intersection, there will be two points X, and the two lines AX will bisect the angle A internally and externally. Also, in this case, in order to prevent the possibility of the construction failing, the lengths AB and AC should both be chosen less than half that of a complete line. The proof is by showing that $XB = XD$, since

$$X\hat{B}D = E\hat{B}A - D\hat{B}A = C\hat{D}A - B\hat{D}A = X\hat{D}B.$$

(2) *To draw a perpendicular to a given straight line.** (Fig. 2.)

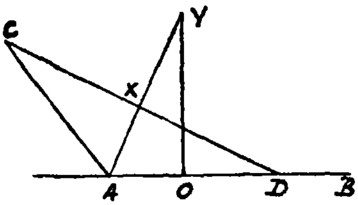

Fig. 2.

Let AB be the given line. Draw another line AC, and cut off AD from AB equal to AC. Bisect the angle CAD by a line meeting CD in X. Cut off AO from AB equal to AX, and AY from AX equal to AD. Join YO; then YO is perpendicular to the given line AB.

(3) *To draw a perpendicular to a given line from a given point on it.* (Fig. 3.)

Let AB be the line, and C the point. Draw a perpendicular YOZ to AB, by (2), meeting it in O, and make OY equal to OZ. Join ZC, YC; and produce YC to X. Bisect the angle ZCX, by (1); then the bisector is the required perpendicular.

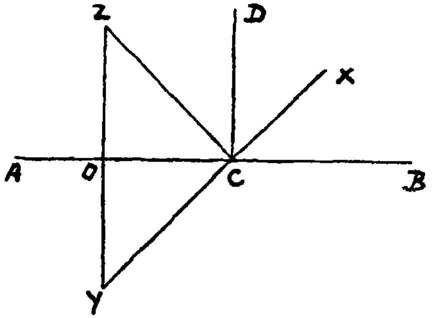

Fig. 3.

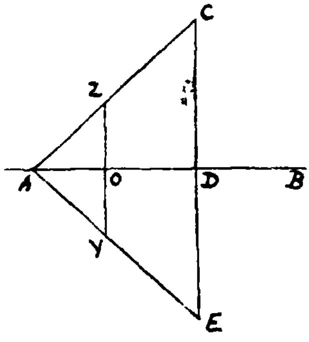

Fig. 4.

If straight lines are closed, a simpler construction can be given, viz.: draw two perpendiculars to AB, by (2), cutting in P, and join CP. This applies whether C is on or off the line.

(4) *To draw a perpendicular to a given line from a given point without it.* (Fig. 4.)

Let AB be the line, and C the point. Join CA. If CA, and also CB, are perpendicular to AB, then every line through C is perpendicular to AB. If CA is not perpendicular to AB, draw a perpendicular to AB from a point O on it, by (3), so as to cut AC in a point Z. This can always be done by choosing O sufficiently near to A. Produce ZO to Y, making *OY* equal to *OZ*. Join *AY*, and produce it to *E*, making *AE* equal to *AC*. Join *CE*; then *CE* is the required perpendicular.

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* This construction, and a similar one, were supplied to the writer by two of his pupils, H. J. Higgs and R. H. J. Sasse.

One perpendicular can always be drawn from *C* to *AB*, and not more than one, unless every line through *C* is perpendicular to *AB*.

(5) *To bisect a given segment.* (Fig. 5.)

Let *AB* be the segment. Draw *AC*, *BD* perpendicular to *AB* *on opposite sides*, and make *AC* equal to *BD*. Then *CD* cuts the segment *AB* at its middle point *O*.

If straight lines are closed, *C* must not be a point such that every line through *C* is perpendicular to the line *AB*, and *C* must not be at the second point of intersection of the lines *AC*, *AB* if there is one. The proof is by first showing that $OC=OD$; and then showing that if *OA* is not equal to *OB* a perpendicular can be drawn from *C* to *AB* different from *CA*, which is impossible. An open segment *AB* has always two *opposite sides*, although a closed straight line may not have two sides.

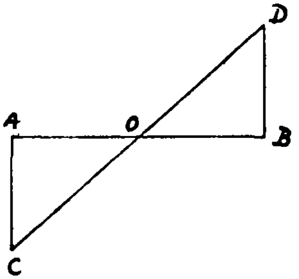


Fig. 5

(6) *To draw a line making with a given line an angle equal to a given angle.* (Fig. 6.)

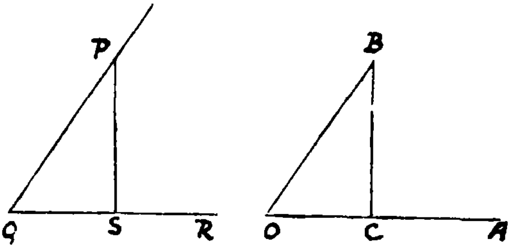


Fig. 6.

Let *OA* be the given line, and *PQR* the given angle. From a point *P* on *QP* draw *PS* perpendicular to *QR*. From *OA* cut off *OC* equal to *QS*. Draw a perpendicular *CB* to *OA*, and make *CB* equal to *SP*. Join *OB*; then *AOB* is the required angle.

F. S. Macaulay.

NOTE ON THE ADDITION FORMULAE OF TRIGONOMETRY.

Proofs of the addition formulae for angles of any magnitudes are sometimes rendered invalid by the way in which arguments are founded on a diagram which can represent only one of the many possible cases. When the two angles under consideration are represented in the usual manner as having a common vertex, the only angles in the diagram whose values it is safe to represent by formulae, if complete generality is to be preserved, are angles having this same vertex. The introduction of other angles may be avoided by a restatement of the definitions of sine and cosine.

At the outset some remarks about the specification of an angle suggest themselves.

(1) When we speak of the angle between two *lines* we mean the angle between two *directions*; it is therefore desirable, even in the most elementary teaching, to postulate that the trigonometrical line, that is a line regarded as an arm of an angle, shall always mean a directed line, a line with an arrow-head.

(2) The merely arithmetic idea of an angle is sufficient in elementary Trigonometry, and it is unwise to suggest to the beginner that an angle described in the clockwise sense must be regarded as negative; the table-book tells us the value of the sine of an angle without asking in which sense the angle is supposed to be described, and the learner ought to be led to take the same point of view. In elemententry Trigonometry and in practical work the measure of an angle is an arithmetical quantity, subject to arithmetic addition and subtraction; the idea of algebraic addition, with the implied idea of a negative angle, is confusing to the beginner and may with advantage be postponed.

(3) Two directed lines are the arms of an infinite series of angles. When we speak of the angle between two lines we mean one of these infinitely numerous angles, and we must know which. The manner of obtaining this knowledge does not concern us at present, but it may be remarked here that in the most frequent application, namely to the triangle, our choice is conventionally directed to the interior angles. The essential thing is that we know what angle is to be discussed before we discuss it; the method of indicating an angle in a diagram by means of an arc of a small circle is, of course, most valuable.

Definitions.

Either arm of an angle may be called the *first* arm, and the angle may be regarded as described by a line starting from the *first* arm and revolving *through the angle* to the *second* arm. The sense in which the revolution takes place depends on which arm we select as the first arm. In order to define the sine and the cosine a choice must be made of the order in which the sides are to be taken, but the choice is quite arbitrary.

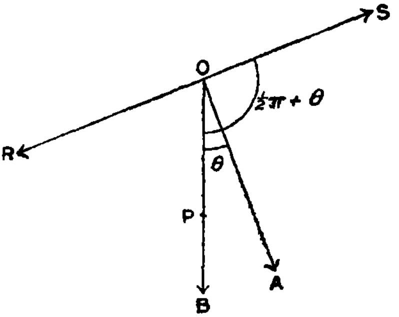
Let OA, OB be the arms of an angle θ, and let OA be chosen as the first arm. Of the two directed lines through O at right angles to OA let OR be that which is such that the right angle AOR and the angle θ would be described in the same sense by a revolving line starting from OA; the directed line OR may be called the *right-angle arm* corresponding to the angle θ. But it will perhaps be more convenient to call OA the *cosine arm*, OB the *second arm*, and OR the *sine arm* of the angle θ. Of course if OB were taken as cosine arm we should get a different sine arm.

Now measure a unit distance OP from O along the second arm OB of the angle, in the direction of the arrow belonging to OB. The *cosine* of θ is defined to be the projection of OP on the cosine arm OA; the *sine* of θ is defined to be the projection of OP on the sine arm OR; projections are to be estimated algebraically, with reference to the arrows on OA and OR.

These definitions have the advantage that they make no mention of 'right,' 'left,' 'up,' 'down,' 'clockwise' or 'counterclockwise.' It follows from them that if OQ be any length, positive or negative, measured along the second arm of an angle θ, its projections on the cosine arm and sine arm respectively are algebraically equal to $OQ \cos \theta$ and $OQ \sin \theta$.

To find the sine and cosine of $\frac{1}{2}\pi+\theta$.

Let AOB be the angle θ, and let OA be taken as cosine arm. Let OR be the corresponding sine arm, and let RO be produced to S. Then the angle SOB is $\frac{1}{2}\pi+\theta$, and we may take OS for its cosine arm. Thus θ and $\frac{1}{2}\pi+\theta$ have the same second arm OB. Let P be in OB at unit distance from O.

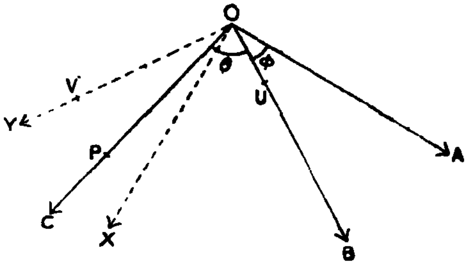


Then OA is the sine arm of $\frac{1}{2}\pi+\theta$, and is the cosine arm of θ. Hence $\cos\theta$ and $\sin(\frac{1}{2}\pi+\theta)$ are both equal to the projection of OP on OA, and so are equal to one another.

Again OR is the sine arm of θ, and OS is the cosine arm of $\frac{1}{2}\pi+\theta$, and these are in opposite directions. The projection of OP on OS is *minus* the projection of OP on OR: hence $\cos(\frac{1}{2}\pi+\theta)= -\sin\theta$.

The Addition Formulae.

Let a straight line start from coincidence with OA and revolve about O always in the same sense. Let OB be its position when it has described an angle ϕ, OC its position when it has described $\phi+\theta$, OX its position when it has described $\frac{1}{2}\pi$, and OY its position when it has described $\frac{1}{2}\pi+\phi$.



Then we may take OA, OB, OX for cosine arm, second arm, and sine arm respectively of the angle ϕ or AOB; we may take OB, OC, OY for cosine arm, second arm, and sine arm respectively of the angle θ or BOC; and we may take OA, OC, OX for cosine arm, second arm, and sine arm respectively of the angle $\phi+\theta$ or AOC.

Let P be at unit distance from O on OC, and let OU, OV be the projections of OP on OB and OY; of course UP and OV are parallel and in the same direction.

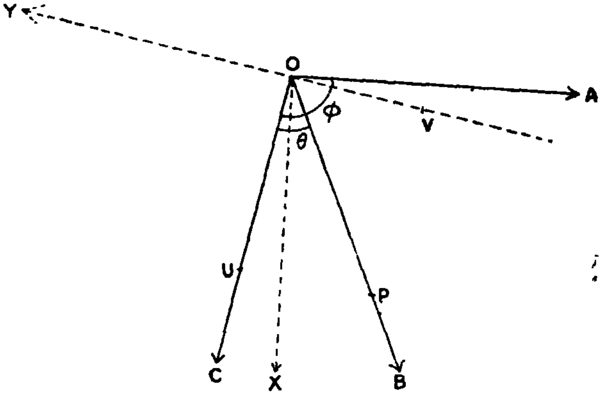
The cosine of $\phi+\theta$ is the projection of OP on OA; this is the same as the sum of the projections of OU and UP (or OV) on OA. But OU is $\cos\theta$, and is estimated positively along the second arm OB of the angle ϕ whose cosine arm is OA ; so the projection of OU on OA is $\cos\theta\cos\phi$. And OV is $\sin\theta$, and is estimated positively along the second arm OY of the angle $\frac{1}{2}\pi+\phi$ whose cosine arm is OA ; so the projection of OV on OA is $\sin\theta\cos(\frac{1}{2}\pi+\phi)$.

Hence
$$\cos(\phi+\theta)=\cos\theta\cos\phi+\sin\theta\cos(\tfrac{1}{2}\pi+\phi)$$
$$=\cos\theta\cos\phi-\sin\theta\sin\phi.$$

The sine of $\phi+\theta$ is the sum of the projections of OU and OV on OX. Now OX is the sine arm of the angle ϕ along whose second arm lies OU, and also of the angle $\frac{1}{2}\pi+\phi$ along whose second arm lies OV. Hence
$$\sin(\phi+\theta)=OU\sin\phi+OV\sin(\tfrac{1}{2}\pi+\phi)$$
$$=\cos\theta\sin\phi+\sin\theta\cos\phi.$$

The Subtraction Formulae.

Let a straight line start from coincidence with OA and revolve about O, always in the same sense. Let OB be its position when it has described an angle $\phi-\theta$, OC its position when it has described an angle ϕ, OX its position when it has described $\frac{1}{2}\pi$, and OY its position when it has described $\frac{1}{2}\pi+\phi$.



Then we may take OA, OB, OX for cosine arm, second arm, and sine arm of the angle $\phi-\theta$ or AOB ; we may take OA, OC, OX for cosine arm, second arm, and sine arm of the angle ϕ or AOC; and we may take OC, OB, and the line OY *reversed* as cosine arm, second arm, and sine arm of the angle θ or BOC.

Let P be at unit distance from O along OP, and let OU, OV be the projections of OP on OC and OY respectively. Then OU is $\cos\theta$; but OV, being estimated positively along OY which is the reversed sine arm of θ, is $-\sin\theta$.

The cosine of $\phi-\theta$ is the projection of OP on OA ; this is the same as the sum of the projections of OU and UP (or OV) on OA. But OU is measured along OC, the second arm of the angle ϕ whose cosine arm is OA ; so the projection of OU on OA is $OU\cos\phi$, or $\cos\theta\cos\phi$. And OV is measured along OY, the second arm of the angle $\frac{1}{2}\pi+\phi$ whose cosine arm is OA ; so the projection of OV on OA is $OV\cos(\frac{1}{2}\pi+\phi)$ or $-\sin\theta\cos(\frac{1}{2}\pi+\phi)$. Hence
$$\cos(\phi-\theta)=\cos\theta\cos\phi-\sin\theta\cos(\tfrac{1}{2}\pi+\phi)$$
$$=\cos\theta\cos\phi+\sin\theta\sin\phi.$$

The sine of $\phi-\theta$ is the sum of the projections of OU and OV on OX. Now OX is the sine arm of the angle ϕ along whose second arm lies OU, and also of the angle $\frac{1}{2}\pi+\phi$ along whose second arm lies OV. Hence

$$\begin{aligned}\sin(\phi-\theta)&=OU\sin\phi+OV\sin(\tfrac{1}{2}\pi+\phi)\\&=\cos\theta\sin\phi-\sin\theta\sin(\tfrac{1}{2}\pi+\phi)\\&=\cos\theta\sin\phi-\sin\theta\cos\phi.\end{aligned}$$

Negative Angles.

The transition from arithmetic subtraction to algebraic addition is effected by giving to the sine and cosine of a negative angle such meanings as shall make the subtraction formulae particular cases of the addition formulae. It is easily seen that the required convention is simply that when an angle is to be regarded as negative the direction of its sine arm must be reversed.

St. John's College, Cambridge. J. G. LEATHEM.

A SUGGESTED REARRANGEMENT OF THE BOOK-WORK ON SOME ELEMENTARY SERIES.

IN all the English text-books on Analytical Trigonometry, so far as I know, the power-series for $\cos x$ is obtained as the limit, when n tends towards* infinity, of the finite sum for $\cos x$,

$$\cos^n\left(\frac{x}{n}\right)\left[1-\left(1-\frac{1}{n}\right)\frac{x^2}{2!}t^2+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)\frac{x^4}{4!}t^4-\ldots\right],$$

where $t=\left(\tan\frac{x}{n}\right)\Big/\frac{x}{n}$ and the number of terms is equal to the integer next greater than $\frac{1}{2}n$. The power-series for $\sin x$ is obtained similarly; the two series and the method of finding them being both due to Euler.

Now it appears to me that this method is subject to certain drawbacks, when introduced in a *first* course on Analytical Trigonometry. For there are two ways of calculating the limit; the harder (but accurate) one as given in books such as Chrystal's *Algebra* or Hobson's *Trigonometry*; and the easier (inaccurate) method of the older text-books, which has unfortunately been reproduced in some new texts.†

The weakness in the inaccurate form of the proof is found in the assumption that the product

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{2r-1}{n}\right)$$

has the limiting value 1, for *all* values of r which do not exceed $\frac{1}{2}n$ (see for instance pp. 109, 110 of Lachlan and Fletcher's book just quoted). The falsity of this assumption is almost self-evident, when thus stated; but the assumption is, nevertheless, nearly always swallowed by the beginner without a qualm.‡

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* It is perhaps worth while to enter a plea for the use of the phrase, *from the very beginning*, in dealing with limits. The phrases "the limit when n *is* ∞," "the limit when $n=\infty$" can do nothing but harm.

† Castle's *Manual of Practical Mathematics* (Macmillan, 1903); Lachlan and Fletcher's *Elements of Trigonometry* (Arnold, 1904). I refer to these as examples, merely; in many other respects both books strike me as good.

‡ I have found a little numerical calculation very useful in convincing the unbeliever that there *is* an assumption involved. Thus with n even, and $r=\frac{1}{2}n$, we find the product to be less than $1/10^3$ when $n=10$; less than $1/10^{10}$ for $n=30$; less than $1/10^{20}$ for $n=50$.

On the other hand the accurate evaluation of the limit is often found hard to grasp, the essential points of the proof being obscured to a certain extent by the details of the algebra. In my opinion the real difficulty arises from the fact that a *double* limit is introduced in the course of the proof; and that an appeal is made to what is really the principle of *uniform convergence* (in a somewhat disguised form). For what is done amounts to this:—The finite sum for $\cos x$ is divided into two parts, the first m terms $\sigma(m, x, n)$ say, and the remainder $\rho(m, x, n)$. It is then proved that if x lies within certain bounds, $\rho(m, x, n)$ can be made numerically less than any arbitrarily prescribed fraction ϵ, by choice of m *only however great n may be* (provided that $n > 2m$). In other words $\rho(m, x, n)$ is proved to converge to zero *uniformly* for *all* values of x, n, subject to the conditions just mentioned. Next, the infinite series $S(x)$ is divided into the first m terms $S(m, x)$ and the remainder $R(m, x)$; and it is proved that for the *same* value of m, $R(m, x)$ is also numerically less than ϵ. Finally, m having been now *fixed*, it is proved that N can be found so that when $n > N$, the difference between $\sigma(m, x, n)$ and $S(m, x)$ is also less than ϵ in numerical value. Hence $\cos x - S(x)$ is numerically less than 3ϵ, provided $n > N$, but $\cos x - S(x)$ is *independent* of n, and consequently $\cos x - S(x) = 0$.

Now this skeleton proof is, for practical purposes of teaching, further complicated by certain algebraic inequalities which must be used in finding the greatest possible values of ρ, R and $\sigma - S$. And, so far as my experience goes, I have found it sufficiently hard to explain the idea of uniform convergence, even when the simplest examples are used, from which all algebraic difficulties have been removed; and it would therefore seem unreasonable to expect a beginner to follow a proof which depends on uniform convergence and involves certain further difficulties of a purely algebraic character.

For the reasons just explained, I have been led to abandon the use of these limits in a first course; and I am inclined to think that other teachers may have felt the same difficulties. My solution of the problem is to give a short course on the elementary Differential Calculus before touching on the sine and cosine power-series. Having explained that when a (*continuous*) function of x has a positive differential coefficient, the function increases with x, I take the following cases:

$$\text{(i)}\quad \frac{d}{dx}(x - \sin x) = 1 - \cos x,$$

$$\text{(ii)}\quad \frac{d}{dx}[\cos x - (1 - \tfrac{1}{2}x^2)] = x - \sin x,$$

$$\text{(iii)}\quad \frac{d}{dx}[\sin x - (x - \tfrac{1}{6}x^3)] = \cos x - (1 - \tfrac{1}{2}x^2),$$

$$\text{(iv)}\quad \frac{d}{dx}[(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4) - \cos x] = \sin x - (x - \tfrac{1}{6}x^3),$$

and so on.

Now, in (1), $1 - \cos x$ is positive and consequently $x - \sin x$ increases with x; it is a continuous function and vanishes for $x = 0$; thus $x - \sin x$ is positive for positive values of x. Apply this to (ii); and by a similar argument $\cos x - (1 - \frac{1}{2}x^2)$ is positive for positive values of x; and this combined with (iii) shows that $\sin x - (x - \frac{1}{6}x^3)$ is positive: passing to (iv), $(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4) - \cos x$ is positive; and so on. Thus we have established that $\sin x$ lies between x and $(x - \frac{1}{6}x^3)$; $\cos x$ between $1 - \frac{1}{2}x^2$ and $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

Clearly the number of terms may be increased to any extent; and we may either introduce the infinite series at once, or leave the results in the form of bounds within which $\sin x$ and $\cos x$ are contained.*

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*For the purpose of numerical calculation the second form is often more useful than the series.

A similar process can be used in connection with arc tan x; for we have

$$\frac{d}{dx}\left[\left(x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5\right)-\text{arc tan}\,x)\right]=1-x^2+x^4-\frac{1}{1+x^2}=\frac{x^6}{1+x^2},$$

$$\frac{d}{dx}\left[\text{arc tan}\,x-\left(x-\tfrac{1}{3}x^3\right)\right]=\frac{1}{1+x^2}-(1-x^2)=\frac{x^4}{1+x^2},$$

and so, since both of these differential coefficients are positive, we can prove that* arc tan x lies between $(x-\frac{1}{3}x^3)$ and $(x-\frac{1}{3}x^3+\frac{1}{5}x^5)$.

By an exactly similar method we can find bounds for e^{-x} and $\log(1+x)$, x being positive; but there is here a logical difficulty, since the differentiation of a^x (and therefore of $\log x$) depends on the limit

$$\lim_{n=\infty}\left(1+\frac{1}{n}\right)^n.$$

The evaluation of this limit by any accurate process leads to difficulties of much the same character as those encountered in the limits for $\cos x$ and $\sin x$. I have *generally found it most convincing* (in a first course) to approximate to the limit by numerical calculation, using 7– or 8– figure tables. Thus we get the approximate values

$$(1+\tfrac{1}{10})^{10}=2{\cdot}593,\ (1+\tfrac{1}{100})^{100}\approx 2{\cdot}705,\ (1+\tfrac{1}{1000})^{1000}\approx 2{\cdot}717,$$

$$(1-\tfrac{1}{10})^{10}=2{\cdot}868,\ (1-\tfrac{1}{100})^{100}\approx 2{\cdot}732,\ (1-\tfrac{1}{1000})^{1000}\approx 2{\cdot}719,$$

which indicate† that the limit is in the neighbourhood of 2·718. Taking the limit as known, the differentiation of a^x, $\log x$, etc., is easy by the methods explained in any good text-book on the Calculus (such as Professor Gibson's).

It may be not out of place to record my opinion that, to a beginner, numerical verification of general theorems on series is often more convincing than the algebraic *proofs*. As a *matter of personal* experience, I found my own faith greatly strengthened by verifying that, to three places of decimals, the sine-series gave 1·000 and the cosine-series 0·000 for $x=1{\cdot}571$; but, as a practical matter, for economy of time smaller values of x will be found preferable, such as $\sin(\cdot5236)=\cdot500$, $\cos(\cdot5236)=\cdot866$, etc.

The sequence of propositions sketched here would, I think, be found to fit in very well with the *Report of the M.A. Committee on Higher School Mathematics* (*Gazette*, vol. 3, July, 1904, p. 52); and it is very likely that the main ideas have already been used by some teachers.‡ The method has the double advantage of giving some elegant illustrations of the elementary rules of the Calculus and, at the same time, of lightening the work in Trigonometry.

In conclusion, I venture to express the opinion that, as a general rule, any work which involves double limits or uniform convergence should be omitted *from the school course; or practically speaking, that "ε-proofs"* should be left out. Such subjects appear to be altogether better suited to a later stage in mathematical training: and the more so because they are entirely unnecessary to anyone but professional mathematicians.§ If this suggestion is accepted, as a matter of course the infinite products for $\sin x$ and $\cos x$ would be omitted (compare the last paragraph of the *Report* quoted above). But, so long as such proofs must be taught, I would strongly recommend the arrangement given in J. Tannery's admirable *Introduction à la Théorié des Fonctions d'une Variable*, a text-book which deserves to be better known in this country than it appears to be.*

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*The angle is taken (as usual in the Calculus) to lie between $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$.

†They also serve to show that the convergence to the limit is very slow, compared with that of the exponential series.

‡Some of the results (perhaps all of them) are given as exercises in Prof. Gibson's excellent *Calculus*; I had, however, obtained them independently before his book was published.

§It is perhaps going too far at present to suggest that the theory of double limits and uniform convergence is essential for a professed mathematician in England; just as, for the present, it seems hopeless to expect even an elementary knowledge of the foundations of geometry from our "up-to-date" Euclids.

T. J. I'a **Bromwich.**

MATHEMATICAL NOTES.

153. [P. 3. b.] *On a Fundamental Theorem in Inversion.*

If two figures F, F' be inverses of each other, their inverses with respect to any pair of corresponding points P, P' are similar.

Demonstration. Let O be the centre, and ρ the radius of inversion of F, F'. Let Q, R be any two points of the figure F; and Q', R' the corresponding points of the figure F'. Also let q, r be the inverses of Q, R with respect to centre P, radius of inversion λ; and q', r' the inverses of Q', R' with respect to centre P', radius of inversion λ'. The theorem will be proved if we show that the ratio of qr to $q'r'$ is constant.

Use the following lemma: If A', B' be the inverses of A, B with respect to a centre O, radius of inversion ρ, then $A'B'/AB=\rho^2/OA \,.\, OB$.

We have by the lemma

$$\frac{qr}{QR}=\frac{\lambda^2}{PQ\,.\,PR}; \quad \text{and} \quad \frac{q'r'}{Q'R'}=\frac{\lambda'^2}{P'Q'\,.\,P'R'};$$

$$\therefore \frac{qr}{q'r'}=\frac{\lambda^2}{\lambda'^2}\cdot\frac{QR}{Q'R'}\cdot\frac{P'Q'}{PQ}\cdot\frac{P'R'}{PR}$$

$$=\frac{\lambda^2}{\lambda'^2}\cdot\frac{\rho^2}{OQ'\,.\,OR'}\cdot\frac{OP'\,.\,OQ'}{\rho^2}\cdot\frac{OP'\,.\,OR'}{\rho^2}$$

$$=\frac{\lambda^2}{\lambda'^2}\cdot\frac{OP'^2}{\rho^2}=\frac{\lambda^2}{\lambda'^2}\cdot\frac{OP'}{OP} \text{ since } \rho^2=OP\,.\,OP'$$

$$=\frac{\lambda^2}{OP}\div\frac{\lambda'^2}{OP'}.$$

Thus the ratio of qr to $q'r'$ is constant and therefore the theorem is proved.

A second demonstration will be given later on. The metrical relation obtained above, namely, that, if λ, λ' be the radii of inversion at P, P' the corresponding lengths in the inverses of F, F' with respect to P, P' are to one another as $\frac{\lambda^2}{OP}=\frac{\lambda'^2}{OP'}$ should be noted, as it is of importance in the applications of the theorem to recent Geometry.

From the theorem it follows that if a figure F undergoes n successive inversions and is thereby converted into F_n, and to the point P in F corresponds the point P_n in F_n, then the inverse of F_n with respect to P_n is similar to the inverse of F with respect to P.

154. [P. 3. b.]. *Continued Inversion by Coaxal Circles.*

An interesting way of obtaining some of Mr. C. E. Youngman's results (p. 7) is by the stereographic projection. Let HK be the points common to the coaxal system. Bisect HK at O, and draw OV equal to OH and perpendicular to the plane of the system. Draw a sphere through V with O as centre. To find the position of P after successive inversions with respect to the circles (A), (B), (C),... we join VP cutting the sphere in Q; bring Q to the position of Q' by successive reflections in the planes through HK perpendicular to VA, VB, VC,...; and then join VQ' cutting the plane AHK in the required position P'. Since successive reflexions in two planes through HK inclined at an angle a are equivalent to a rotation through $2a$ about HK, and since the circles (A) and (B) cut at an angle AVB = the angle between the planes through HK perpendicular to VA and VB, we have at once $[AB]=[CD]$, provided the circles (A) and (B) cut at the same angle as (C) and (D). If (C) or (D) is the line HK we have Mr. R. F. Davis's result. Again, if (A) and (B) are orthogonal $[AB]=[BA]$, for rotations through π and through $-\pi$ about HK are equivalent, etc.

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* It may not be out of place to refer to the fact that the real difficulty in finding the infinite product is not to prove that $x(1-x^2/\pi^2)$, etc. are factors; but to prove that *no factor of the form e^{ax} is present* This remark is due to Stolz, but is not made in any of the English text-books

HAROLD HILTON.

155. [K. 20. a.]. *Definitions of Trigonometrical Ratios, and General Proof of Addition-Theorems for Sine and Cosine.*

The object of this note is to indicate a method of treating the fundamental theory of Trigonometry which, so far as simplicity and completeness are concerned, seems to have considerable merit. No absolute novelty can be claimed for or alleged against it. (See, *e.g.*, Casey's *Trigonometry*.)

We shall use the symbol $\equiv$ as denoting "has the same length, direction and sense as"; and the symbol = will as usual denote both equality in magnitude and likeness in sign.

Prop. I. The projections on a *directed* line Ox of any two directed line-segments of equal length and the same direction, are equal to one another.

Proof. Let AB, CD be two such segments, and ab, cd their projections on Ox.

Let AE and CF drawn ∥ to Ox meet Bb and Dd in E and F. Then the triangles AEB, CFD are equal in all respects, having their corresponding sides in the same directions.

$$\text{Thus} \qquad AE \equiv CF; \qquad \therefore ab \equiv cd.$$

Prop. II. If a directed line meet the directed line Ox in A, and any two segments of the former, starting from A, be taken, then the ratio of projection on Ox to segment is the same for the two segments.

Proof. Let AB, AG be two such segments (which may have like or unlike senses) and Ab, Ag their projections on Ox.

$$\text{Then} \qquad \frac{gA}{Ab}=\frac{GA}{AB}; \qquad \therefore \frac{Ag}{AG}=\frac{Ab}{AB}.$$

Prop III. The ratio of projection on Ox to segment is the same in magnitude and sign for any two segments of any two directed lines that make the same angle a with Ox.

Proof. If LM and PQ be such segments, and the line LM meet Ox in A; and if along this line we lay off segments $AN \equiv LM$ and $AR \equiv PQ$, then the ratio of projection to segment is the same for LM and for AN (Prop. I.) and for AR (Prop. II.) and for PQ (Prop. I.).

Def. I. The ratio of projection on Ox to segment, for any directed segment of a directed line making an angle a with the directed line Ox, is the *cosine of* a.

Note that a directed segment of a directed line is positive or negative according as its sense is the same as that of the line or opposite to it. We have shown in Props. I., II., and III. that this definition of $\cos a$ is complete.

Def. II. The corresponding ratio when the projection is on a line making an angle $+90°$ with Ox is called $\sin a$.

Prop. IV. $\operatorname{Cos}(\alpha \pm 180^\circ) = -\cos\alpha$.

For $\cos(\alpha+180^\circ)$ is the cosine of the angle that a line inclined at α to Ox makes with the line inclined at -180° to Ox, *i.e.* with xO; and the projection of any segment on Ox is $= -$(its projection on xO).

Prop. V. $\operatorname{Cos}(-\alpha) = \cos\alpha$.

The proof is obvious, if we take segments of equal magnitude starting from the same point of Ox.

Cor. 1. $$\begin{aligned}\operatorname{Sin}(-\alpha) &= \cos(-\alpha-90^\circ) = \cos(\alpha+90^\circ) = \cos(\alpha-90^\circ+180^\circ)\\ &= -\cos(\alpha-90^\circ)\\ &= -\sin\alpha.\end{aligned}$$

Cor. 2. $$\begin{aligned}\operatorname{Sin}(\alpha-180^\circ) &= -\sin(-\alpha+180^\circ) = -\cos(-\alpha+90^\circ) = -\cos(\alpha-90^\circ)\\ &= -\sin\alpha.\end{aligned}$$

Cor. 3. $\operatorname{Cos}(\alpha+90^\circ) = \cos(-\alpha-90^\circ) = \sin(-\alpha) = -\sin\alpha$.

Prop. VI. $\operatorname{Cos}(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$.

Let $$X\hat{O}H = \alpha, \quad H\hat{O}P = \beta, \quad H\hat{O}K = 90^\circ;$$
$$\therefore\ X\hat{O}P = \alpha+\beta, \quad X\hat{O}K = \alpha+90^\circ.$$

Let N and M be the projections of P on OH and OK.

Now projection on Ox of OP = projection of ON + projection of NP

$$\begin{aligned} &= \quad ,, \quad ON + \quad ,, \quad OM\\ &= ON\cos\alpha + OM\cos(\alpha+90^\circ) \qquad \text{(by Def. I.)}\\ &= ON\cos\alpha - OM\sin\alpha \qquad \text{(by Prop. V., Cor. 3).}\end{aligned}$$

Again, $ON = OP\cos\beta$; and $OM = OP\cos(\beta-90^\circ) = OP\sin\beta$.

Hence projection on Ox of $OP = OP\cos\alpha\cos\beta - OP\sin\alpha\sin\beta$.

$$\therefore \text{ by Def. I., } \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

Cor. 1. $$\begin{aligned}\operatorname{Cos}(\alpha-\beta) &= \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta.\end{aligned}$$

Cor. 2. $$\begin{aligned}\operatorname{Sin}(\alpha-\beta) &= \cos(\alpha-\beta-90^\circ)\\ &= \cos\alpha\cos(\beta+90^\circ) + \sin\alpha\sin(\beta+90^\circ)\\ &= -\cos\alpha\sin\beta + \sin\alpha\cos\beta.\end{aligned}$$

Cor. 3. $$\begin{aligned}\operatorname{Sin}(\alpha+\beta) &= \sin\{\alpha-(-\beta)\} = \sin\alpha\cos(-\beta) - \cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta + \cos\alpha\sin\beta.\end{aligned}$$

Definitions III., IV., V., VI. :—

Tan α is the ratio $\sin\alpha : \cos\alpha$,
cot α ,, $\cos\alpha : \sin\alpha$,
sec α ,, $1 : \cos\alpha$,
cosec α ,, $1 : \sin\alpha$.

R. F. MUIRHEAD.

REVIEWS.

HERMANN GRASSMANN. **Gesammelte mathematische und physicalische Werke.** II. Band, I. Theil. **Die Abhandlungen zur Geometrie und Analysis.** 1904. (Teubner, Leipzig.)

Hermann Grassmann was born in Stettin in 1809, his father being the mathematical professor at the Gymnasium, and the author of several text-books. On his father's death the son succeeded to his position, and remained in Stettin till his death in 1877.

It may come as a surprise to some that the Stettin schoolmaster's scientific writings are so extensive as to be republished in six substantial volumes. During his lifetime and for some years after his death, little attention was paid to his works, partly owing to their novelty, partly to the obscure style in which they were written. In the progress of mathematical thought the same ideas have been reached by other channels, so that Grassmann's writings have become much more intelligible than they were to his contemporaries, and we can marvel at the prophet who was so much in advance of his time.

The edition of Grassmann's works which is now in course of publication owes its inception to Klein, who, in 1892, persuaded the Royal Philosophical Society of Saxony to authorise the undertaking, and secured the services of Engel as general editor, *assisted by* Luroth, Study, Scheffers and *two sons of the author.*

It will represent the English usage more closely it we translate "Band" by "Part" and "Theil" by "Volume." The first volume contains the "Ausdehnungslehre" of 1844 and the second that of 1862; papers on Geometry and Analysis are collected in the volume under review, and papers on Mechanics and Physics in the fourth. The third part and last two volumes, which have not appeared yet, will contain the Prufungsarbeit on Tides, and matter of a critical and historical character.

Nearly three hundred pages are occupied by papers of a geometrical or algebraical nature, chiefly applications of the methods of the Ausdehnungslehre to the theory of plane curves, in particular, to their linear generation. The next fifty pages contain extracts from a school text-book on Arithmetic, published in 1861, followed by a short extract from one on Trigonometry published in 1865. A list is given of the most important alterations that have been made from the original editions, and this is followed by ample notes of an explanatory nature.

R. W. H. T. HUDSON.

G. HOLZMULLER. **Vorbereitende Einfuhrung in die Raumlehre.** Pp. 123. 1904. (Teubner, Leipzig.)

It is explained on the title page that this little book is published "Im Anschluss an die preussischen Lehrplane von 1901 zur freien Auswahl fur den Anfangsunterricht bearbeitet und mit Anleitungen zum Herstellen von Unterrichts-Modellen versehen." Everyone agrees nowadays that a course of "observational" or "practical" geometry should precede the logical theory; but there is a considerable danger of the reasoning faculty being altogether sacrificed to the cultivation of geometrical intuition. The two are quite distinct and even opposed to each other, and the fact that they are customarily exercised upon the same subject matter introduces great difficulties, although it is a necessary means of supplying sufficient interest.

This book begins with a careful discussion of the elements of space, the ultimate appeal being to intuition. The greater part of the book is devoted to constructions in two and three dimensions. The author rightly observes that the difficulties of "solid" geometry are usually over-estimated. It is only after long years of artificial restriction to the abstract plane that the power of realising space configurations is enfeebled: it should be encouraged while it is still natural.

R. W. H. T. HUDSON.

Leçons élémentaires sur la Théorie des Fonctions Analytiques. EDWARD A. FOUET. Seconde Partie. Pp. 299. 1904. (Gauthier-Villars.)

In this, his second, volume M. Fouet includes the last two chapters of the first book and the first three chapters of Book II. The first chapter contains a general discussion of functions defined by means of differential equations, and includes a short account of the work of MM. Fuchs, Poincaré and Painlevé, in connection with the existence of parametric critical points (pp. 34 *sqq*). M. Painlevé demonstrated that out of equations of the form $y'' = R(y', y, x)$ where R is rational in y', algebraic in y, and analytic in x, he has determined explicitly all those of which the essential critical points are fixed, and that where, in such cases, the general integral is an essentially new uniform function, the equations can be reduced to three simple types. He suggested also the way of discussing equations of higher order.

Painlevé's work in connection with his three types of differential equations has appeared within the last few years (1900-1903), so that there would seem to be furnished a comparatively new field in the discussion of the properties of the new transcendants he has thus defined. M. Fouët, however, fails to give any account of Painlevé's methods, and ignores the work of Picard and of Mittag Leffler dealing with the same subject (*C. R.* vols. CII., CIV., etc. *Acta Math.* vols. XVII., XVIII., etc.). In this first chapter M. Fouët gives certain theorems on the existence of integrals of partial equations, and concludes with theorems connected with Dirichlet's Problem.

The last chapter of Book I. is concerned with functions defined by functional properties, such as, for example, properties of periodicity. Two pages are devoted to the consideration of doubly periodic functions.

In the three chapters of Book II. given in this volume, analytic functions are discussed from the points of view of Cauchy, Weierstrass and Riemann.

M. Fouët's main difficulty has apparently arisen in connection with the arrangement of the work. He is perhaps justified in his attempt to give separate discussions of analytic functions from the three different standpoints, and this attempt necessitates a general account of function theoretic ideas on the lines of that given in Book I. But the effect of his arrangement is a little unfortunate in such cases as the following:—

The discussion of the function $\Gamma(z)$ is divided into two parts, one of which appears in Book I. and the other in Book II. The elementary discussion of infinite products, which is essentially connected with Weierstrass' point of view of function theory, is separated from it by such matter as a general discussion of differential equations, and an account of Dirichlet's problem. This problem itself, which is intimately connected with Riemann's, is not treated in connection with the account of that work, but in connection with differential equations; and an account of the properties of harmonic functions, which should be given in connection with the discussion of Laplace's Equation, is reserved to make an addition to the chapter on Riemann's theory of functions. If M. Fouet intends to discuss in detail such an important branch of his subject as elliptic functions, it would be an advantage to have the account of the whole subject in one part of the treatise. Instead of this he has considered functions possessing an algebraic addition theorem in Vol. I., p. 168, and in Vol. II., pp. 83, 84, while the Weierstrassian σ function is given as a product in Vol. II., p. 188, general discussion being apparently reserved for a later volume.

The work is chiefly interesting for its account of many of the outlying parts of the subject, such as, for example, that of minima surfaces (Vol. II., pp. 285 *sqq.*). It is unfortunate that, owing to the scope of the work, such accounts are necessarily brief.

J. E. Wright.

Leçons sur les séries à termes positifs. Par. É. Borel. Pp. 91. 1902. (Paris, Gauthier-Villars.)

This is a reproduction of 20 lectures given by M. Borel at the *Collège de France* in the session 1900-1901; they have been collected and edited by one of M. Borel's class, M. Robert d'Adhémar. It forms a part of the series of works entitled *Nouvelles leçons sur la théorie des fonctions*, in which have appeared M. Borel's lectures on integral functions and on divergent series.

In chapter 1 we are concerned mainly with the familiar logarithmic criteria of convergence, which M. Borel attributes to Bertrand, though they appear to have been given first by de Morgan (see Chrystal's *Algebra*, vol. 2, historical note to chap. xxvi.). It is not until we reach the middle of chapter 2 that M. Borel introduces us to the idea which forms the central thread of the book; this idea is that of the *croissance* (degree of rapidity of increase) of a function always increases with the independent variable x. For the simple power x^n, the *croissance* is represented by the index n; for the exponential e^x, a symbol ω is introduced, which must [in virtue of the property $\lim\limits_{x=\infty} (x^n/e^x)=0$] be greater than any ordinary index; the symbol ω has also been used by G. Cantor to represent the *transfinite number*, which gives a useful analogy for the exponential *croissance*. The laws of combination of the symbol ω are investigated, and the results are tabulated on p. 47; they agree only partially with the algebraic laws of ordinary numbers.*

---

* *E.g. Multiplication* is defined as in the following examples: ωn is the *croissance* of exp. (x^n), but $n\omega$ is that of $(\text{exp. } x)^n$; ω^2 is that of exp. (exp. x); and so on. Thus $\omega n \neq n\omega$; but $\omega^\alpha\omega^\beta = \omega^{\alpha+\beta}$.

Passing to chapter 4 we find two simple criteria for the convergence of double series (with positive terms) which may be quoted here; taking the series $\Sigma\Sigma v_{mn}$ $(m, n = 1, 2, \dots \infty)$ we have a *sufficient* test of convergence if, after a certain stage, we can say that $v_{mn} < (m+n)^{-(2+p)}$, where $p > 0$; or that $v_{mn} < \Sigma 1/(m^{2+p} + n^{2+p})$. But if $v_{mn} > (m+n)^{-2}$, the series diverges. Similar conditions are then obtained for double integrals, in which the area of integration extends to infinity.

In chapter 5 we take up the question of the *croissance* of a power-series $\Sigma a_n x^n$: it is shown that if $\phi(n) = a_n^{-1/n}$ is an increasing function, of order n^ρ, then the *croissance* of the function defined by the series is $\omega(1/\rho)$; but the inverse problem is much harder.* Another interesting result is given for the case in which $\phi(n)$ *tends to the limit 1 as n tends towards* ∞; *taking the series* Σa_n *to be divergent*, the behaviour of the function $\Sigma a_n x^n$ near $x = 1$ is determined. For instance,

$$\lim_{x=1} [(1-x)^p(1^{p-1}x + 2^{p-1}x^2 + \dots + n^{p-1}x^n + \dots)] = \Gamma(p).$$

The *remainder of the chapter* can hardly be appreciated without a direct reference to the original; the book terminates with a short account of analogous properties of the double series $\Sigma a_{mn} x^m y^n$.

We have no hesitation in recommending those who are interested in modern analysis to add this book to their shelves; it is a worthy successor to M. Borel's earlier text-books; a higher recommendation cannot be given.

T. J. I'a Bromwich.

Théorie élémentaire des séries. Par M. Godefroy. Pp. 266. 1903. (Paris, Gauthier-Villars.)

This book differs entirely in its aims from M. Borel's; as appears both from the title and the method of treatment, it is mainly intended for beginners in the field of analysis. It is somewhat similar to the treatment given in Chrystal's *Algebra* and Hobson's *Trigonometry*, except that the complex variable is nowhere used;† in this respect following the example of the first edition of Stolz's *Allgemeine Arithmetik*. However, M. Godefroy assumes a knowledge of the elements of the Differential Calculus, and so is able to give some theorems which are not mentioned in the treatises named: for example (p. 75) we have the theorem that a linear differential equation is soluble by power-series which converge within the same interval as the power-series which represent the co-efficients. Some of the simpler properties of Bessel's and Legendre's functions (of the first kind) are obtained.

It is something of a novelty to find in a professedly elementary book, a very clear and concise discussion of the Weierstrassian continuous, non differentiable, function

$$\Sigma r^n \cos(a^n \pi x),$$

where $0 < r < 1$ and a is an an odd integer such that $ar > 1 + \frac{3}{2}\pi$. According to a footnote to this article, M. Poincaré has remarked that a hundred years ago a function such as this would have been thought an outrage on common sense; but we fear that in England a much more recent date might be assigned.

M. Godefroy concludes his book with 50 pages on the gamma-function; for the most part, these articles appear to be reprints of those in his earlier work bearing that title (see *Math. Gazette*, March, 1902).

As a whole *the work should be found extremely useful by teachers who have* to lecture on the elementary theory of infinite series; and I have made considerable use of it in my course of lectures on the subject.

T. J. I'a Bromwich.

Theorie und Praxis der Reihen. By C. Runge. Pp. 266. 1904. (Goschen, Leipzig.)

In this admirable monograph Professor Runge has devoted his attention to the practical use of series at the expense of the general theory and all its details. The following is a summary of the five chapters into which the book is divided.

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* For the function $\phi(n)$ may be changed so as to increase quite irregularly without in any way affecting the *croissance* of the series.

† This will be regarded as an advantage by those teachers who think it well to begin the study of power-series before introducing the complex variable.

Chap. I. Series with constant terms. Convergence. Operations on series with extension to series with complex terms. Chap. II. Uniform convergence (with graphical illustrations). Power series. Recurrence. Reversion. Integration and Differentiation of uniform series. Cauchy's theorem on the integration of a uniform function of a complex variable with applications. Cylindrical functions of one variable. Interpolation series. Chap. III. A simple treatment of Fourier's series. Michelson and Stratton's Harmonic Analyser. Chap. IV. Infinite Products. Circular and theta functions. Chap. V. Development in series of functions of more than two variables. Extension of Taylor's series and others. Spherical functions.

A useful and interesting little volume.

CORRESPONDENCE.

To the Editor of the "MATHEMATICAL GAZETTE."

CIVIL SERVICE COMMISSION,
BURLINGTON GARDENS, W., 23/9/04.

DEAR SIR,—Once there was no law in nature, and every thing did what seemed good in his own eyes. But some things said, "This is not good. Behold the civilization of man. Let us also have laws." And a conference was held, and to it came representatives of all forms of matter and energy. And at the conference they said, "There is a good and just man called Isaac Newton. He would make just laws and not severe." And they asked him. And he made laws just and not severe, and wrote them in the Latin tongue. And the conference accepted the laws, and said all things must obey, even things mental and spiritual, for (they said) why did they not come to the conference? So all nature obeyed.

In course of time it happened that none knew the Latin tongue, and none knew the law to obey it. So the two chief priests read Latin diligently, and put the laws into the common tongue, and wrote commentaries on them so precise that no thing dared say he did not understand. And there was peace.

Such a myth would explain our recent teaching of mechanics. The Committee of the Mathematical Association deserve our thanks for rebelling and founding the science again on experiment. But here and there their view might be made clearer. Suggestion 22 might be thought to support the old method. And it seems a pity, after obtaining experimental results in Part A, to discard in Part C all but the parallelogram of forces and to deduce all others from this in a way that would gladden the heart of Euclid.—Yours faithfully DAVID MAIR.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

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The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I.-IX. Edited and published by E. Hendricks, M.A., Des Moines, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by Artemas Martin, M.A. (Erie, Pa.)]

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed.) Edited by J. D. Runkle, A.M.

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Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

BOOKS, ETC., RECEIVED.

A New Trigonometry for Schools. Part I. By W. G. Borchardt and A. D. Perrot. Pp. vi, 237, xii, xxii. 1904. (Bell.)

Plane Surveying (including Levelling). By S. M. Barton. Pp. viii, 255. 6s. 1904. (Heath.)

Trigonométrie (Second Cycle). By E Borel. Pp. viii, 198. 2 fr. 50. 1904. (Colin.)

Exercises in Arithmetic, Oral and Written. By C. M. Taylor. Part III. Pp. viii, 132, 20. 1s. 6d. 1904. (Arnold.)

Elements of Trigonometry. By R. Lachlan and W. C. Fletcher. Pp. viii, 164. 2s. 1904. (Arnold.)

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Diagrammes et Surfaces thermodynamiques. By J. W. Gibbs. Pp. 85. (Scientia) 2 frcs. 1903. (Gauthier-Villars.)

Géométrographie ou art des Constructions géométriques. By E. Lemoine. Pp. 87. (Scientia) 2 frcs. 1902. (Gauthier Villars.)

Cours de Géométrie Analytique. By A. Tresse and A. Thybaut. Pp. iii, 550. 12 frcs. 1904. (Armand Colin.)

A Treatise on Hydromechanics. Part. I. *Hydrostatics.* By W. H. Besant and A. S. Ramsey. Pp. viii, 264. 6th edition. 1904. (Bell & Sons.)

Longmans' Senior Arithmetic. By T. F. G. Dexter and A. H. Garlick. Pp. x, 554. 1904. 4s. 6d. (Longmans.)

Plane Trigonometry. By J. M. Taylor Pp. iv, 171. 1904. 3s. 6d. (Ginn.)

New School Arithmetic. Part I. By C. Pendlebury and F. E. Robinson. Pp. xv, 206, xxi. 1904. 2s. 6d. (Bell.)

Practical Geometry for Beginners. By V. Le Neve Foster and F. W. Dobbs Pp. viii, 96. 2s. 6d. 1904. (Macmillan.)

Bulletins della Associazione "Mathesis." Edited by PROF. R. BETTAZZI. No. 5. 1903-04. Pp. 59-68.

Bulletins de la Société Physico-Mathematique de Kasan. 2nd Series, Vol. XIII. Pp. 120, 184. 1903.

Plane Geometry, Practical and Theoretical. By J. S. MACKAY. Pp. viii, 236. 2s. 6d. 1904. (Chambers.)

Key to Godfrey and Siddons' Geometry. By E. A. PRICE. Pp. 172. 5s. net. 1904. (Cam. Univ. Press.)

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A School Geometry. Part VI. By H. S. HALL and F. H. STEVENS. Pp. iv, 347-442, iv. 1s. 6d. 1904. (Macmillan.)

A New School Arithmetic. Part II. By C. PENDLEBURY. Pp. vi, 207-468, xliv. 2s 6d. 1904. (Bell.)

Preparation of the Child for Science. By M. E. BOOLE. Pp. 157. 2s 1904. (Clarendon Press.)

Solutions of Examples in Hall's Graphical Algebra. By H. S HALL and H. E. BEAVER. Pp. 59. 1904. (Macmillan.)

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A Proposito d'un Buon Libro d'Astronomia. By BETTINO CARRARA S.J. Pp 11. 1904. (Artigianelli, Monzo.)

The World Time Finder. (Diagram Co., 27 Victoria Rd., S.W.)

Columbia University Bulletin of Information. Mathematical and Physical Science. 1904-1906.

Report on Teaching of Elementary Plane Geometry. (Assist. Masters' Assocn.)

A Radianometer. 1s. *Leather Snap Case of Instruments.* 7s. 6d. (Lilley & Sons, Forest Works, Dalston, N.E.)

Arithmetical Examples. By W. G. BORCHARDT. 2nd Edition, pp. viii., 247. 3s. 1904. (Bell.)

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THE

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NOTE ON THE "METHOD OF THE ARITHMETIC MEAN" AS APPLIED TO RATES OF INCREASE.

IT is known to all students of physics that when a number of different observations have been made of the same constant quantity, the best result is, in general, obtained by taking the arithmetic mean of the observed values.

If the observed quantity instead of being constant is increasing at a uniform rate, and we require to determine this rate of increase, we should naturally expect that the proper course to follow would be to find the rates of increase of the quantity in successive equal intervals of time and take their arithmetic mean.

Suppose, then, that $y_0, y_1, y_2, \ldots y_n$ are the observed values of the quantity, the interval between every two consecutive observations being equal to t, and there being altogether n intervals. The rates of increase in the several intervals are

$$\frac{y_1-y_0}{t},\ \frac{y_2-y_1}{t},\ \ldots\ \frac{y_n-y_{n-1}}{t}.$$

The arithmetical mean of these is

$$\frac{y_n-y_0}{nt},$$

and thus depends only on the initial and final values and not on the intermediate ones. It is, however, quite contrary to all ideas of common sense that this result should be regarded as giving the most probable value, since its accuracy depends wholly on two measurements and no weight is attached to the intermediate measurements.

If we plot the observed values on squared paper the above result would mean that the rate of increase was determined by the straight line joining the first and last points plotted. It might, however, be found that the intermediate points were very nearly in a straight line, and that, owing to considerable errors in the first and last values, the first and last points were far from being in a line with the rest. It is obvious that in such cases the method would give a bad result.

This difficulty can be overcome by applying the method of Least Squares in the following manner. Let it be assumed that the correct value of the observed quantity y at time t is given by the formula

$$y = a+bt,$$

where a, b are constants, b denoting the required probable rate of increase. Then by the method of Least Squares, if times are measured from the first observation, we have to make

$$(y_0-a)^2+(y_1-a-ib)^2+(y_2-a-2ib)^2+\dots+(y_n-a-nib)^2$$

a minimum by the variation of a and b. Differentiating with respect to a and b, we have

$$(y_0-a)+(y_1-a-ib)+(y_2-a-2ib)+\dots+(y_n-a-nib)=0,$$

$$(y_1-a-ib)+2(y_2-a-2ib)+\dots+n(y_n-a-nib)=0.$$

These give

$$(n+1)a+\tfrac{1}{2}n(n+1)ib=y_0+y_1+\dots+y_n, \quad\dots\dots\dots\dots(1)$$

$$\tfrac{1}{2}n(n+1)a+\tfrac{1}{6}n(n+1)(2n+1)ib=y_1+2y_2+\dots+ny_n. \quad\dots\dots\dots\dots(2)$$

Eliminating a, we have

$$\tfrac{1}{6}n(n+1)(n+2)ib=ny_n+(n-2)y_{n-1}+(n-4)y_{n-2}+\dots-(n-2)y_1-ny_0.$$

This formula gives b, the required probable rate of increase. We may write it in the form:

$$b=\frac{n(y_n-y_0)+(n-2)(y_{n-1}-y_1)+(n-4)(y_{n-2}-y_2)+\dots}{\tfrac{1}{6}n(n+1)(n+2)i}, \quad\dots\dots\dots\dots(3)$$

and it will be seen that, unlike the previous formula, it takes account of intermediate as well as extreme values. But it still remains to find a simple interpretation showing why and wherefore the formula takes this particular form. This interpretation may be given in the following rule:

Take all the different pairs of observed values that can be found by combinations of the $n+1$ observations taken two at a time. Add together the total increases and divide by the sum of the time intervals which these increases represent.

We may verify this rule by exhibiting the differences and the time intervals in the following tabular form:

Differences.	*Intervals.*	
y_n-y_{n-1}	i	
y_n-y_{n-2}	$2i$	
y_n-y_{n-3}	$3i$	
…………	…………	
…………	…………	
y_n-y_0	ni	$\tfrac{1}{2}n(n+1)i$
$y_{n-1}-y_{n-2}$	i	
$y_{n-1}-y_{n-3}$	$2i$	
…………	…………	
…………	…………	
$y_{n-1}\quad y_0$	$(n-1)i$	$\tfrac{1}{2}(n-1)ni$
…………	…………	
…………	…………	
y_1-y_0	i	i
$ny_n+(n-2)y_{n-1}+\quad-(n-2)y_1-ny_0$		$\tfrac{1}{6}n(n+1)(n+2)i$

We may also state the rule as follows: The total increases of value determined from each of the $\frac{1}{2}n(n+1)$ pairs of observed values taken two at a time are of equal weight.

If, instead of taking all possible pairs of observations, we take only the pairs formed by the first and last, the second and last but one, and so on, the total increases in the corresponding intervals are $y_n - y_0$, $y_{n-1} - y_1$ and so on, and the lengths of the intervals are ni, $(n-2)i$, ... With this arrangement the weights of the several intervals are proportional to the lengths of these intervals, as is evident from equation (3).

Equation (1) gives

$$a + \tfrac{1}{2}nib = \frac{y_0 + y_1 + y_2 + \dots + y_n}{n+1}$$

or the most probable value of the quantity y at the middle of the period of observation is the arithmetic mean of the observed values, as it evidently should be.

To take a numerical example let us calculate the average annual rate of increase of the quantity of pig iron produced by Great Britain during the period 1891-1901, the annual quantities, in million tons, being as follows (from Whitaker's *Almanack*):

7·4, 6·7, 7·4, 7·3, 7·6, 8·6, 8·7, 8·5, 9·4, 8·8, 7·8.

The process may be written thus:

	Increase.	No. of years.		Sums.	Corresponding No. of years.
$d_1 = y_{10} - y_0$	0·4	10	d_1	= 0·4	10
$d_2 = y_9 - y_1$	2·1	8	$d_1 + d_2$	= 2·5	18
$d_3 = y_8 - y_2$	2·0	6	$d_1 + d_2 + d_3$	4·5	24
$d_4 = y_7 - y_3$	1·2	4	$d_1 + d_2 + d_3 + d_4$	5·7	28
$d_5 = y_6 - y_4$	1·1	2	$d_1 + d_2 + d_3 + d_4 + d_5$	6·8	30
			$5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5$	= 19·9	110

The mean rate of increase given by the formula is 19·9 ÷ 110 = ·181 million tons per annum; whereas if the increase were calculated by dividing the difference between the first and last values by the number of years, the mean rate of increase would be only ·04 million tons. This is a very good example of the use of the method. There is very little difference between *the first and last values of the observed quantity, but the remaining values* show that the production was considerably greater as a rule towards the end than towards the beginning of the decade. Here, then, we have a case in which the difference between the initial and final values fails to afford an efficient measure of the general increase.

The tabular working of the above example also shows a useful method of calculating such quantities as $(1a + 2b + 3c + 4d \dots) \div (1 + 2 + 3 + 4 \dots)$. The numbers in the right-hand half are the sums of the corresponding numbers in the left-hand half, added up thus: "0·4 and 2·1 are 2·5 and 2·0 are 4·5 and 1·2 are 5·7 and 1·1 are 6·8." A similar method of adding up may frequently be found of use in obtaining average of marks of candidates in examinations, when the distribution of the candidates over the various percentages is known.

George H. Bryan.

University College of North Wales,
Bangor, Wales, 11*th October*, 1904.

THE DEVELOPMENT OF GEOMETRICAL METHODS.*

In order to give an adequate account of the progress of Geometry in the century that has just come to a close, it is important to cast a rapid glance over the condition of Mathematical Science at the beginning of the nineteenth century.

We know that during the latter portion of his life, Lagrange, somewhat weary of the researches in Analysis and Mechanics, which, however, assured for him immortal glory, took up the study of Chemistry (which he described as almost as easy as Algebra), of Physics, and also devoted himself to philosophical speculation. We almost always see a similar state of mind at certain periods in the lives of the greatest men of science. From the fresh ideas of their fertile youth, ideas which they utilise to the full in the field of their labours, they have derived all that they have any right to expect: their task is done: their mental activity is impelled towards entirely new subjects. And this impulse, be it understood, had a quite peculiar force in the days of Lagrange. The field of research opened to geometers by the discovery of the Infinitesimal Calculus appeared to be almost exhausted. It really seemed that when a few more or less complicated differential equations were integrated, and a few more chapters were added to the Integral Calculus, the very limits of Science would be attained. Laplace had completed his explanation of the system of the world, and was laying the foundations of Molecular Physics. New paths were opening for the experimental sciences, and all was ready for the astonishing development they were to receive during the nineteenth century. Ampère, Poisson, Fourier, and Cauchy himself, the creator of the theory of imaginaries, were entirely taken up with the application of analytical methods to Mechanics and Molecular Physics; they seemed to believe that apart from the new domain of which they were making so rapid a survey, the limits of Theory and Science were definitely fixed.

From the end of the eighteenth century, the subject for which we must claim the name of modern Geometry had been in a large measure contributing to the renaissance of Mathematical Science in all its branches. It had offered to research a new and fertile path, and, in particular, it had shewn in the most striking and successful manner, that general methods are not everything in Science, and that even in the simplest subject there is much that may be done by the ingenious and inventive mind. The beautiful geometrical demonstrations of Huyghens, Newton, and Clairaut were forgotten or neglected. The fine ideas introduced by Desargues and Pascal had never been developed and seemed to have fallen on barren soil. Carnot, by his *Essai sur les Transversales* and his *Géométrie de Position*, and Monge, more especially by his beautiful theories on the generation of surfaces, had pieced together a chain the links of which had seemed to be severed. Thanks to them, the conceptions of Descartes and Fermat, the inventors of Analytical Geometry, assumed once more, in the Infinitesimal Calculus of Leibniz and Newton, the position they had lost and which they should never have ceased to occupy. As Lagrange said of Monge, "ce diable d'homme" will make himself immortal by his Geometry. And in fact, not only did Descriptive Geometry play its part in coordinating and perfecting the processes employed in all the arts, "in which precision of form is a condition of success as well as of excellence for labour and its products," but it made its appearance as the graphical translation of a general and purely rational Geometry the fertility of which has been shewn by researches both numerous and important. At the side of his *Géométrie Descriptive* we must not forget to place his other master-piece, the *Application de l'Analyse à la Géométrie*. Nor should we forget that to Monge we owe the idea of lines of curvature, and the elegant integration of the differential equations of those lines in the case of the ellipsoid, which it was said was a matter of envy to Lagrange. We must lay stress on this character of the whole of the work of Monge. The man who reconstructed what we call modern Geometry has shewn us from the outset, and this has perhaps been forgotten by his successors, that the alliance between Analysis and Geometry was useful and fruitful, and that perhaps this alliance was a condition of success for both these branches of Mathematics.

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* An address by M. Gaston Darboux, of the *Institut de France*, Sept. 24, 1904, given at the Congress of Science and Art, St. Louis, U.S.A. The author has very kindly given permission for the publication of this interesting address in the *Gazette*.

This distinguished mathematician is Permanent Secretary to the Académie des Sciences; Honorary Dean of the Faculté des Sciences. In addition to numerous Memoirs he has published: *Leçons sur la Théorie des Surfaces et les Applications géométriques du Calcul Infinitésimal.* 4 vols. in 8°. 1887-1889-1894-1896 (Gauthier-Villars). *Coordonnées curvilignes.* 1 vol. in 8°. 1898 (Gauthier-Villars).

II.

To the school of Monge we owe many geometers: Hachette, Brianchon, Chappuis, Binet, Lancret, Dupin, Malus, Gaultier de Tours, Poncelet, Chasles, etc. Among these, Poncelet ranks first and foremost. He passed over everything in the works of Monge connected with Cartesian Analysis or with Infinitesimal Geometry; he devoted his attention exclusively to the development of the germs contained in the purely geometrical researches of his illustrious predecessor. He was captured by the Russians at the passage of the Dnieper, and interned at Saratoff. During the period of enforced leisure which followed he turned his attention to the proof of the principles which he developed in his *Traité des Propriétés projectives des Figures*, published in 1822, and in the great memoirs on polar reciprocals and harmonic means which date back to about this time. It may therefore be said that modern Geometry was born at Saratoff. Returning to the ideas which had been neglected since the days of Pascal and Desargues, Poncelet simultaneously introduced homology and polar reciprocals, thus at the outset throwing into relief the ideas which Science had been maturing for 50 years.

Poncelet's methods were not enthusiastically received by French analysts, being in a sense opposed to the processes of Analytical Geometry. But their importance was so great and their novelty so striking that it was not long before they were thoroughly examined. Poncelet alone had discovered the principles; on the other hand, several geometers almost simultaneously began to study them from every point of view and succeeded in deducing from them the essential results which they implicitly contained.

At this time, Gergonne was the editor of a periodical which is of inestimable value to us in the history of Geometry. The *Annales de Mathématiques*, published at Nîmes from 1810 to 1831, had been for more than fifteen years the only journal in the world exclusively devoted to mathematical research. Gergonne was in many respects a model editor for a scientific journal: but he had the defects of his qualities. He collaborated, often against their will, with the authors of the memoirs which were sent to him: he altered their papers, and sometimes made them say more or less than they had intended. However that may be, he was greatly struck with the originality and the scope of Poncelet's discoveries. A few simple ways of transforming figures were already known in Geometry: homology had even been used in the plane; no one but Poncelet had extended it to space, and no idea seemed to exist of the wealth and fertility of the method. Besides, all these transformations were *ponctuelles*, point was made to correspond to point. By the introduction of polar reciprocals, Poncelet became an inventor of the highest rank; he gave the first example of a transformation in which to a point corresponded something else than a point. Every method of transformation enables us to multiply our theorems, but the method of polar reciprocals had the advantage of bringing into correspondence two entirely different propositions. This was essentially something new. To throw this into relief Gergonne invented the plan, which has since been adopted with so much success, of writing correlative propositions in two columns. His also was the idea of substituting for Poncelet's demonstrations, which required the assistance of a curve or surface of the second order, the famous principle of duality, the significance of which, though somewhat vague at first, was sufficiently cleared up by the discussions which took place on the subject between Gergonne, Poncelet, and Plücker.

Bobillier, Chasles, Steiner, Lamé, Sturm, and many others, whose names escape me, assiduously collaborated with Poncelet and Plücker on the *Annales*. When Gergonne was appointed Rector of the Académie de Montpellier in 1831, his publication of the journal was interrupted. But his success in creating a new form of mathematical literature, and the taste for research which he had done so much to develop, had already begun to bear fruit. Quetelet had just started the Belgian *Correspondance mathématique et physique*. Crelle in 1826 had published the first sheets of the famous journal which bears his name, and in which are to be found memoirs by Abel, Jacobi, and Steiner. A considerable number of formal treatises also appeared at this time, in which the principles of modern geometry were being expounded and developed in a masterly manner.

First, there was in 1827 the *Barycentrische Calcul* of Möbius, a really original work, remarkable for the depth of its conceptions and the lucidity and rigour of its exposition. In 1828 appeared Plücker's *Analytisch-geometrische Entwickelungen*, the second part of which was published in 1831. These were very soon followed by the same author's *System der analytischen Geometrie* (Berlin, 1835). In 1832 Steiner published his *Systematische Entwickelung der Abhängigkeit der geometrischen Gestalten von einander*, and in the next year appeared his *Geometrische Constructionen ausgeführt mittelst der geraden Linie und eines festen Kreises*, in which was proved by the most elegant examples a proposition of Poncelet's relative to the use of a single circle in geometrical constructions. Finally, in 1830, Chasles sent to the Académie de Bruxelles, which by a happy inspiration had offered a prize for the best essay on the principles of modern geometry, his celebrated *Aperçu historique sur l'origine et le développement des méthodes en Géométrie*. This was followed by the *Mémoire sur deux principes généraux de la Science: la dualité et l'homographie*, which was not published until 1837.

Time would fail me were I to endeavour to offer an adequate appreciation of these noble works, or on such an occasion as this to give an account of each of them. Such an attempt would lead us to a new verification of the general laws of the development of science. When the moment is ripe, when the fundamental principles are recognised and enunciated, nothing can check the march of ideas; the same or nearly the same discoveries are made almost simultaneously, and in all parts of the world. But although I shall not venture on a discussion of this kind, which would be of little use, and might give but little pleasure, it is important that I should try to throw some light upon a fundamental distinction in the tendencies of the great geometers who, about 1830, gave to Geometry an impulse hitherto unknown.

III.

Some, like Chasles and Steiner, who devoted the whole of their lives to researches in Pure Geometry, drew a distinction between what they called *synthesis* and *analysis*; and, adopting in their general scope, if not in their detail, the tendencies of Poncelet, endeavoured to found an independent theory, a rival to the Cartesian Analysis.

Poncelet did not rest content with the inadequate resources furnished by the method of projections. He therefore imagined the famous *principle of continuity*, which gave rise to such long discussions between him and Cauchy. The principle is an excellent one when properly enunciated, and may be of the greatest service. Poncelet made the mistake of refusing to present it as a simple consequence of Analysis; Cauchy, on the other hand, would not recognise that his own objections, valid no doubt in the case of certain transcendental figures, lost their force in the applications made by the author of the *Traité des propriétés projectives*. Whatever opinion may be held on the subject of such a discussion, it at least shewed, in the clearest manner, that Poncelet's geometrical system rested on an analytical basis, and we know besides, from the unfortunate publication of the Saratoff manuscripts, that the principles which served as a foundation for the *Traité* were established by the aid of the Cartesian Analysis. Poncelet had abandoned Geometry for Mechanics, in which his work had a preponderating influence. Chasles, his junior, for whom had been created in 1847 a chair of Higher Geometry in the *Faculté des Sciences de Paris*, set to work to build up a geometrical theory entirely independent and autonomous. This he expounded in two works of the highest importance, his *Traité de Géométrie Supérieure* (1852), and the *Traité des Sections Coniques*, unfortunately left unfinished, the first part alone appearing in 1865.

In the preface to the former he very clearly indicates the three fundamental points which enable the new theory to participate in the advantages of analysis, and seem to him to mark some progress in the cultivation of the science. They are:

(1) The introduction of the principle of signs, which simplifies both enunciations and proofs, and gives to the analysis of Carnot's transversals all the scope of which they are susceptible.

(2) The introduction of imaginaries, which supplements the

principle of continuity, and furnishes demonstrations as general as those of analytical geometry.

(3) The simultaneous demonstration of correlative propositions, *i.e.*, propositions which correspoud in virtue of the principle of duality.

Chasles thoroughly discusses homography and correlation in his work, but in his exposition he systematically avoids the use of the transformation of figures. These, he thinks, cannot supplement direct proofs, because they mask the origin and real nature of the properties obtained by their means. There is something in this criticism, but the march of science shews us that it errs on the side of severity. Even if it often happens that transformations, when employed without discrimination, uselessly multiply the number of theorems, yet we must not forget that they often assist us to a better realisation of the propositions to which they are applied. Did not the use of Poncelet's projection, which led to the fruitful distinction between projective and metrical properties, make us recognise the great importance of this anharmonic ratio, the essential property of which is to be found in Pappus, and the fundamental rôle of which did not appear in modern geometry until fifteen centuries later?

The introduction of the principle of signs was not the novelty Chasles supposed when he wrote his *Traité de Géométrie Supérieure*. Möbius in his *Barycentrisches Calcul* had already followed up a *desideratum* of Carnot's, and had used signs in a very wide and precise manner, defining for the first time the sign of a segment and even of an area. Later on he succeeded in extending the use of signs to lengths which are not measured off on the same straight line, and to angles which are not formed around the same point. In addition, Grassmann, whose mind presents so many points of analogy with that of Möbius, was necessarily compelled to use the principle of signs in the definitions which serve as a basis for his remarkably original method of discussing the properties of extension.

The second characteristic assigned to his system of geometry by Chasles is the use of imaginaries. Here his method was really new, and he was able to illustrate it by examples of the greatest interest. Admiration will always be awarded to the beautiful theories he has left us on the confocal surfaces of the second degree, in which every known property and many new properties, as varied as they are elegant, are derived from the general principle that these surfaces are inscribed in the same developable which is circumscribed to the circle at infinity. But Chasles only introduced imaginaries by their symmetrical functions, and was therefore unable to define the anharmonic ratio of four elements when they cease to be real, in whole or in part. If only he could have established the idea of the anharmonic ratio of imaginary elements, a formula which is to be found in the *Géométrie Supérieure* (p. 118, new edition) would have immediately enabled him to give the beautiful definition of an angle as the logarithm of an anharmonic ratio—a definition which enabled Laguerre, my lamented colleague, to find the long-sought, complete solution of the problem of the transformation of those relations which contain both angles and homographic and correlated segments.

IV.

While Chasles, Steiner, and later, von Staudt, were endeavouring to build up a theory that would be a rival to Analysis, and were, as it were, erecting altar against altar, Gergonne, Bobillier, Sturm, and Plucker in particular, were perfecting the Cartesian Geometry, and were constructing an analytical system in some measure adequate to the discoveries of the geometers. To Bobillier and Plucker we owe what is known as the method of *Abridged Notation.* In the last volumes of Gergonne's *Annales* are to be found a few really original pages from the pen of the former. Plücker had begun to develop the method in his first volume, very soon to be followed by a series of works in which the foundations of modern geometry are deliberately established. To the same investigator we owe tangential and trilinear coordinates, used in homogeneous equations, and also the canonical forms, the validity of which is recognised in the fruitful but sometimes deceptive method known as the *enumeration of constants.* All these timely discoveries were to infuse new blood into the Cartesian Analysis; they enabled it to give their full significance to those conceptions which the so-called *synthetic* geometry had not completely grasped. Plucker—and with his name it is only fair to couple that of Bobillier, carried off by an untimely death—must be regarded as the first to familiarise us with those methods of modern Analysis, in which the use of homogeneous coordinates enables us to treat simultaneously, without the reader's knowledge, as it were, a figure, and all the figures that are deduced from it by homography and correlation.

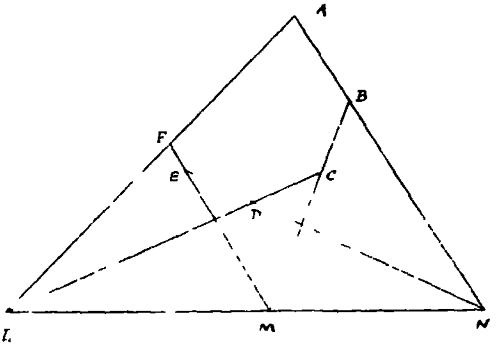
(*To be continued.*)

MATHEMATICAL NOTES.

156. [L^2. 7; 14. a.] *To prove by Pascal's theorem that the straight lines meeting three non-intersecting straight lines generate a conicoid, i.e. a surface every plane section of which is a conic.*

Let P_1, P_2, P_3; p_1, p_2, p_3 be two sets of non-intersecting straight lines in space, each line of one set meeting all of the other set. Let them meet any plane, not containing one of them, in the points A, C, E, B, D, F. Let AF, CD; FE, BC; ED, AB meet respectively in L, M, N. Denote the intersections P_1p_2, P_2p_3, P_3p_1 by X, Y, Z. Now X, Y, L lie in one straight line for they will all be found in both the planes P_1p_3 and P_2p_2, and therefore lie on their intersection.

Similarly Y, Z, M lie in the planes P_3p_3 and P_2p_1, and X, Z, N lie in the planes P_3p_2 and P_1p_1. Therefore L, M, N are the points in which the sides of the triangle XYZ meet the plane of $ABCDEF$, and are therefore in one straight line, so that by Pascal's theorem A, B, C, D, E, F lie in a conic. Since five points determine the conic, we may consider F, and therefore p_3, as variable, and any straight line meeting P_1, P_2, P_3 traces out the conic $ABCDE$.



W. H. Blythe.

157. [P. 3. b. α.] *To find the relation between two maps of the same contour on the gnomonic projection.*

Let l be any line touching at A a sphere whose centre is O, and let m be the parallel line through O. Let ξ be the projection from O on the sphere of any curve α on the tangent plane p at A. Let ζ be the reflection of ξ in a plane through m making an angle θ with the plane mA and cutting p in the line n. Let η be the reflection of ζ in the plane mA. Then η is obtained from ξ by rotating ξ about m through an angle 2θ. Let β, γ be the projections of η, ζ on p. Then α, β are two different gnomonic projections of the same spherical curve. Draw VAW perpendicular to l in the plane p cutting n in W, such that VOW is a right angle. Then the line joining corresponding points of α and γ evidently passes through V and is divided harmonically by V and its intersection with n. Hence β is the reflexion in l of the harmonic homologue of α, V being the centre, and n the axis of homology.

Since the theorems given in the *Math. Gazette*, Jan. 1904, p. 383, and March, 1904, p. 7, on successive inversion with respect to coaxial circles can be proved by aid of the "relation between two maps of the same contour on the stereographic projection" (*Math. Gazette*, May 1904, p. 33; Oct. 1904, p. 88), we see that in the theorems referred to we may replace "inversion with respect to one of a system of coaxial circles" by "the operation of taking the harmonic homologue of a figure," "the axis of homology being one of a system of parallel lines and the centre its pole with respect to a fixed circle." An independent proof may be readily given.

Harold Hilton.

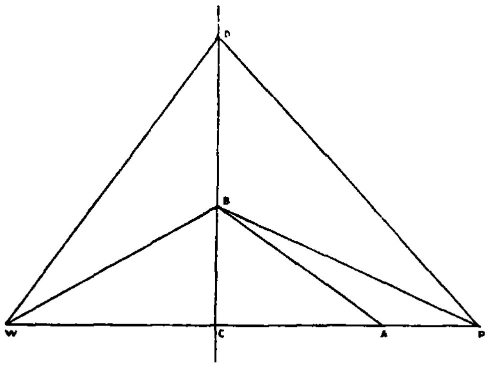
158. [P. 3. b.] *On a Theorem in Inversion.*

Since two points P, P' on a sphere which are reflexions of each other in the plane of a great circle s are projected stereographically into two points inverse with respect to the projection of s, the theorem 153 (p. 88) becomes: the stereographic projections of two figures on the sphere which are reflexions of each other in the plane of s from P and P' respectively are enantiomorphous —which is obvious (cf. 154, p. 88). If we use the gnomonic projection instead of the stereographic, we have: If $a(F)$ denote the harmonic homologue of any figure F, A being the centre of homology and the polar of A, with respect to a given circle the axis of homology, then $\pi(F)$ and $\pi'(F')$ are enantiomorphous, where $a(F)=F'$, $\beta(A)$ is at infinity, $\beta(P)=P'$, and AB passes through the centre of the given circle (cf. Note 157).

HAROLD HILTON.

159. [P. 3. b. α.] *A note on the gnomonic projection.*

Let $B, C, D, P, \ldots$ be the projections from O on the tangent plane at A of the points $B', C', D', P', \ldots$ on a sphere whose centre is O. (B may be conveniently marked with a × or a o according as the angle BOA is acute or obtuse.) Draw $PACW$ perpendicular to BD meeting BD in C; let $WC^2=AC^2+AO^2$ and $CA \,.\, AP=AO^2$. Then the angle $BWD=B'OD'=$ the angle between the great circles $B'P'$, $D'P'$, since $B'P'$, $D'P'$ are quadrants, and any point on BD is equidistant from O and W. These facts enable us to measure the angle and sides of any spherical triangle whose gnomonic projection is given. They also enable us to solve graphically any spherical triangle; for the gnomonic projection of any spherical triangle on the tangent plane at a vertex can be readily found when the three sides, or the two sides meeting at that vertex and any angle are given.



Moreover the projection enables us to prove the usual formulae of spherical trigonometry. For example: since

$$\sin BAC=\frac{BC}{BW}\div\frac{BA}{BW};\ \cos BAC=\frac{AC}{AO}\div\frac{AB}{AO};\ \tan BAC=\frac{CB}{CW}\div\frac{CA}{CW};$$

we have the following formulae for a spherical triangle right-angled at C: $\sin A=\sin a \,.\, \operatorname{cosec} c$; $\cos A=\tan b \,.\, \cot c$; $\tan A=\tan a \,.\, \operatorname{cosec} b$.

HAROLD HILTON.

160. [L¹. 3. a.] *On the Equation to the axes of the general conic.*

The points on the ellipse $x^2/a^2+y^2/b^2=1$ the normals at which cointersect in a given point (ξ, η) are determined as the intersections of the ellipse with a definite rectangular hyperbola $a^2\xi/x-b^2\eta/y=a^2-b^2$. When $\xi=0$, $\eta=0$ this R.H. degenerates into the two axes of the ellipse.

Similarly the points on the general conic the normals at which cointersect in a given point (ξ, η) lie on the R.H.

$$(ax+hy+g)/(x-\xi)=(hx+by+f)/(y-\eta).$$

Hence making (ξ, η) the centre of the curve, the equation to the axes of the conic is

$$(ax+hy+g)(y-F/C)=(hx+by+f)(x-G/C).$$

R. F. DAVIS.

161. [**R. 4. d.**] *Graphic Construction for the Central Axis of two Non-intersecting Forces.*

Let P, Q be the forces, α the angle between their directions, d the shortest distance between them.

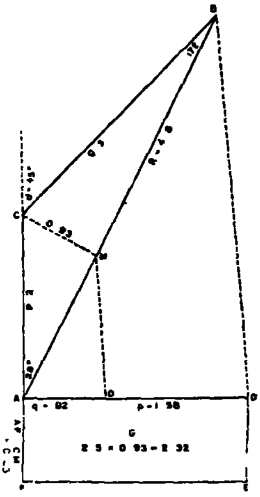
Draw AC, CB representing P, Q in magnitude and direction, so that $ACB = 180° - \alpha$.

Then AB represents in magnitude and direction the resultant force R. Draw CM perpendicular to AB.

In any direction measure off AD equal to the shortest distance d. Divide AD in the ratio of AM to MB, and let O be the point of section.

Then AO, OD are the segments into which the shortest distance is divided by the central axis, OA being the distance from Q (which we shall call q) and OD the distance (p) from BP.

The moment of the resultant couple is represented by the area of a rectangle $ADEF$ whose base AD is the shortest distance d, and whose height AF is equal to CM.



The construction in the figure was made for $P=2$, $Q=3$, $\alpha=45°$, $d=2{\cdot}5$, the lengths in the original figure being measured in inches.

On measurement the following rough results were found:

$$AB = 4{\cdot}8 \qquad \angle CAB = 28 \qquad \angle CBA = 17,$$
$$AO = 0{\cdot}92 \qquad OD = 1{\cdot}58 \qquad CM = 0{\cdot}93,$$

leading to the following conclusions:

Resultant force = 4·8 units, making angles 28° with the force of 2 units and 17° with the force of 3 units.

Central axis is 1·58 inches from the former and 0·92 inches from the latter force, measured along the shortest distance.

Moment of resultant couple $= 2{\cdot}5 \times 0{\cdot}93 = 2{\cdot}32$. G. H. BRYAN.

REVIEWS.

Etudes sur les quantités mathématiques; grandeurs dirigées—quaternions. By CLARO CORNELIO DASSEN. Pp. vi, 136. 5 francs. 1904. (A. Hermann, Paris.)

This volume consists of two parts, the concept of quantity, pp. 1-34, and directed quantities, pp. 35-112, followed by appendices on analysis based uniquely and entirely on the notion of the whole number, on angles and arcs, "dits imaginaires," and on space of four dimensions and quaternions.

Part I. contains an interesting account of the history of the fundamental symbols of arithmetic and of various systems of scales of notation. The associative, distributive and commutative laws of arithmetical operations are examined in considerable detail. Chap. I., Part II., is occupied with a discussion of the laws of combination of "quantites dirigées sur un droit ou en deux sens," and here as elsewhere throughout the work there are numerous references to what Clerk Maxwell called the "archaeology" of the subject. The second chapter deals with directed quantities in a plane, and contains critical examinations of the various interpretations that have been assigned to the square root of minus one. Directed quantities in space are considered in Chap. III. (pp. 86-112), which is chiefly occupied with quaternions, though a page or two are devoted to the Ausdehnungslehre, and the Icosian calculus is mentioned. In introducing the conception of the multiplication of directed quantities in space, the author says, "Si l'en veut donc étendre la théorie de la multiplication aux vecteurs dirigés dans l'espace, on est obligé de renoncer à la conservation de certaines propriétés

combinatoires de cette opération; mais comme—il ne faut pas l'oublier—les définitions et les suppositions que l'on fera doivent être subordonnées à leur utilité pratique, il faut se garder de laisser à la fantaisie le choix des hypothèses; au contraire, on tâchera de conserver les propriétés les plus importantes, celles qui se prêtent aux combinaisons des calculs; ces propriétés sont sans contredit l'uniformité, l'associativité et la distributivité par rapport à l'addition, car la commutativité et l'existence d'un module ne sont pas nécessaries pour le mécanisme du calcul." Would that the "fantaisie" of the writers in the *Encyklopadie der math. Wiss.* had been controlled by such common sense; the preservation of the associative law would lead to the extinction of most of the numerous family of systems of vector analysis. C. J. JOLY.

Leçons Élémentaires sur la théorie des Groupes de Transformations. Par G. VIVANTI. Traduites par A. BOULANGER. Pp. 296. (Paris, Gauthier-Villars.) 1904.

Nothing is stated in the title or preface of this book as to its previous appearance in Italian, or as to whether it has been translated under the supervision of the author. In the absence of information to the contrary it is assumed to be an authorised reproduction in French of the course delivered by Professor Vivanti at Messina in the years 1897-8, and at the time lithographed.

It must have been a valuable course of lectures, calculated to give hearers a true notion of the fundamentals and salient features of Lie's great theory, in practically unmodified order and form, and to encourage those who could to complete the study of the volumes from which it was in effect culled. As a printed treatise it will be of the same utility to a larger circle of students, and especially so in a new language. But it cannot be considered a work of an enterprising or original character. The straightforwardness of the procedure by which it has been constructed is described in the following extract from the preface: "Nous parcourons les deux premiers volumes du grand ouvrage de Lie, en cherchant à montrer les grandes lignes et à fixer les points fondamentaux de la nouvelle doctrine; nous consacrerons aussi quelque pages à l'application des vues de Lie aux Equations differentielles ordinaires." To this should have been added that in the pages last referred to the method has been the same as in the others, with Scheffers now for guiding authority upon Lie's views. It is the way of a painstaking tutor, who goes through a voluminous standard treatise with a red pencil, marks the sections suitable for a first reading, and then reduces the marked portions to a manuscript for the benefit of his pupils. He often thinks, rightly enough, that this would be useful printed, but is generally fastidious.

The first 132 pages of the book are devoted to the general theory of groups of transformations, in other words to Lie-Engel's first volume. The extracts made from this are judicious and fairly ample. Then come 61 pages on differential equations of the first order; and, lastly, 99 pages on contact transformations.

Granted that an editing of a first course taken from Lie is to be welcomed, granted also that a first course for editing has been chosen with discretion, the question remains whether the variations of expression introduced into the proofs of propositions selected do or do not tend in the direction of simplicity. Most of the variations here are merely matters of paraphrase and notation. The rest are of two kinds: (1) removals of extreme elaboration and painfully minute attention to rigour and completeness, and (2) introductions of pieces of preliminary analysis, in cases where Lie's arguments had directly used, as known or proved at an earlier stage, what could hardly be deemed part of the common knowledge of beginners or had been excluded. These are to be commended. Consequently the book, regarded as an unambitious one, is quite good of its sort.

A good preparatory work, however, may be made much better by the introduction into it of numerous elementary illustrative examples. Here there are practically none, except about a score (taken without exception from Scheffers) in the differential equations section. E. B. ELLIOTT.

Vorlesungen über Technische Mechanik. Zweite Band. **Graphische Statik.** By Dr. AUG. FOPPL, Professor at the Munich Technical Institute. Pp. xii and 472. (Teubner.) 2nd edition. 10 marks.

Professor Foppl's four volumes of Vorlesungen are devoted to elementary mechanics, graphic statics, elasticity, and dynamics respectively, and aim at giving a student some idea of modern advanced methods, in their application to certain technical problems, without employing very elaborate mathematical apparatus.

In the present volume, after a brief sketch of the rise of graphical methods, the author explains the scope of his work. He does not confine himself to the purely graphical standpoint of Cremona, in whose hands the subject is as it were a branch of projective geometry, but uses the methods of the calculus freely when convenient. On the other hand, the author declines to discuss in the lecture-room questions as to the material, design, and mode of erection of particular structures which ought to be dealt with at the works.

One object which the author keeps in view from the first is that the student should become accustomed to deal with forces in space. Thus, for instance, instead of a bare statement of the parallelepiped of forces, a graphical solution is given of the problem of resolving a force acting at a point into components acting along three given straight lines through the point, specified by their plan and elevation.

An excellent account of Bow's method of dealing with frames, the usual link polygon constructions for finding centres of gravity and moments of inertia, and the leading propositions on the composition and resolution of forces in space follow in order.

The features, however, which present most novelty to an English reader occur in the chapter on the elastic deformation of frames. In this chapter a simple graphical method (which appears to be little known here) of investigating the elastic displacements of the joints of a frame is given, which is said to be due to M. Williot (*Seyrig: Statique Graphique*). The question is one of considerable importance in the case of a structure of the cantilever type. Many readers will recollect the difficulty which arose in getting the central girder of the Forth Bridge into position.

An application after Maxwell (cf. Routh's *Statics*, vol. i., pp. 161-162) of the principle of virtual work to finding the stresses in the bars of a statically indeterminate frame gives rise to some interesting examples.

Special attention is given to those frames for which, though they are not statically indeterminate, the ordinary method of drawing a Bow's diagram fails and some special artifice becomes necessary. An instance of the latter class, of which the so-called Belgian roof is the most familiar example, occurred in the Mathematical Tripos this year. Those exceptional cases in which a frame is self-strained though it contains no redundant bars are also considered. They were first noticed by Professor Crofton (*Proc. Lon. Math. Soc.*, vol. x.), who showed that a frame consisting of a Pascal's hexagon with its three diagonals might be self-strained. Professor Crofton alludes to the surprise which Pascal's hexagon must feel at finding itself involved in a discussion on applied mechanics.

Professor Foppl's work abounds in interesting applications, and no review would do justice to it which failed to call attention to these and to the superlatively excellent diagrams.

C. S. JACKSON.

Elements of Plane Surveying. By SAMUEL MARX BARTON, Ph.D., Professor of Mathematics in the University of the South. Pp. viii and 255. (Boston: D. C. Heath & Co.) 1904.

The making of a survey for an underground railway beneath the streets of a great city raises some delicate problems, of which an interesting account will be found in the *Proceedings of the Institute of Civil Engineers* (vol. clvi., p. 376), but apart from such peculiar cases a topographical survey, as distinguished from a valuation of a farm or estate, is in England nowadays, owing to the existence of the large scale ordnance maps, a comparatively rare event.

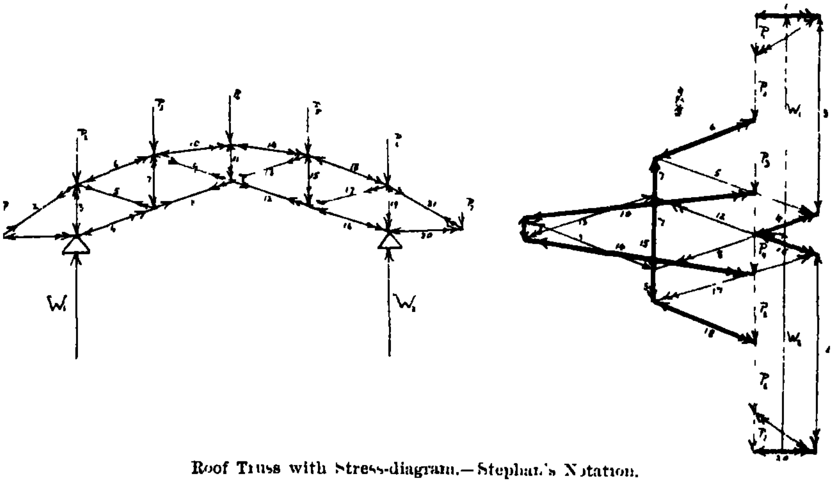
In the United States it appears, however, that the making of a survey is still an important branch of the professional duties of a surveyor, and in the present work the author has described the principal methods and instruments used in making an ordinary survey of a small tract of ground. The methods of geodetic surveying, and the photographic processes employed by Colonel Laussedat, do not come within the author's plan, but in a future edition a reference to Sir Howard Grubb's "ghost" sights and levels (*Trans. R.S., Dublin*, vii. 385) might prove useful.

An interesting substitute for the compass is described in Saegmuller's solar attachment to a theodolite, which automatically carries out the process given in books on navigation of finding compass error by observation, thus dispensing with the necessity for any computation. An auxiliary telescope is set, by means of the known latitude and the known declination of the sun, into such a position relative to the principal telescope that when the auxiliary telescope is on the sun the principal telescope, and with it the zero line on the bottom plate of the theodolite, are necessarily in the meridian. The true N. and S. line thus determined enables the observer to read off the true bearing of any object. The same instrument furnishes a simple means of finding the time by observation. A table annexed gives the effect of an error of one minute in latitude or declination on the resulting azimuth. A chapter is devoted to modern methods of "stadia" surveying, and the work concludes with a set of tables, accompanied by some notably careful and intelligible directions for their use. C. S. JACKSON.

Die Technische Mechanik. First Part: Mechanics of Rigid Bodies. By P. STEPHAN. Pp 344. (Teubner, Leipzig.)

In the preface of this work the author states that his book is primarily intended for use in Schools of Machine Design. Statics is treated of before Dynamics, partly in order that the most simple technical questions, such as friction, may be considered as early as possible, but chiefly in order that too many new ideas and definitions may not be brought forward at the commencement.

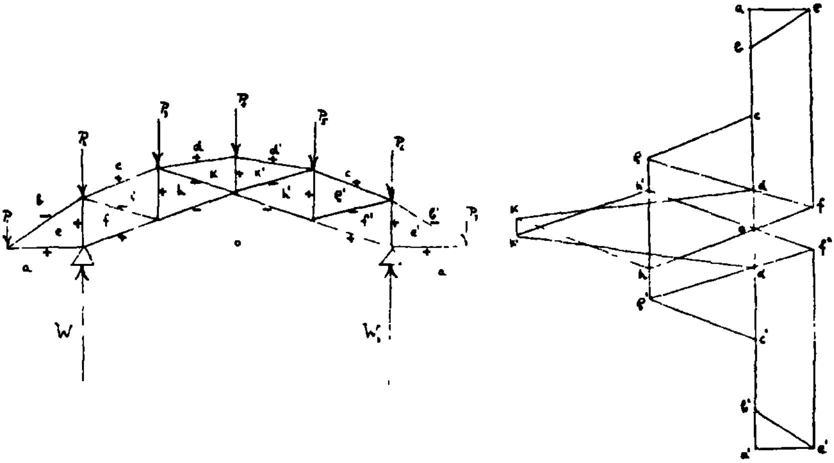
The use of the Calculus is avoided throughout the volume, and "dodges" for circumventing its use are freely indulged in. Thus two pages (66 and 67) are devoted to the determination of the Centre of Gravity of a parabolic segment. As another illustration mention might be made of the proof of the formula $T=T_0e^{\mu a}$ for the tension of a rope coiled round a rough surface. The proof, which we do not recollect having seen before, is ingenious and obtains the well-known result as the limiting value of $T_0\left(1+\mu\frac{a}{n}\right)^n$ when n is increased indefinitely (pp. 169-171). Proofs of this kind are going more and more "out of fashion" in English text-books, and in this we are wholly in sympathy. In "dodging" the Calculus and attempting to find an easier path, we find that at each step the difficulties get greater and greater. While in mastering the Calculus all the difficulties lie in the initial stages, and when the fundamental notions are fairly grasped the subsequent path is an easy one. When all is said and done, the "dodge" is nothing more or less than the Calculus itself,



Roof Truss with Stress-diagram.—Stephan's Notation.

though it is dressed up so as to look like a piece of commonplace Algebra or Trigonometry.

It is a matter of the greatest surprise that Bow's notation for force-diagrams of framed structures has not been universally adopted in foreign text-books. The confusion produced by the multiplication of arrow heads in the bars of the roof truss on p. 204 seems almost hopeless. The stress-diagram as given by Stephan for this roof is here reproduced, and for comparison the same diagram with Bow's notation. Comment seems superfluous. The struts and *ties are distinguished by positive and negative signs placed beside the bars of the frame.* Stephan, on the other hand, uses thick lines in the force-diagram to denote the struts. If the force diagram is actually to be used for the measurement of the stresses, this can only lead to faulty results. There would not be the same objection to indicating the corresponding bars of the frame-diagram by thick lines, and, in some ways, this is preferable to using the more usual positive and negative signs. This, however, might be criticised on the score that a bar might be a strut for one system of loading and a tie for another system.



Roof Truss with Stress-diagram.—Bow's Notation.

About 200 examples are worked out in the text, many of them showing a pleasing originality and freshness which is sadly to seek in our ordinary school text-books. Nearly all the examples are accompanied by clear diagrams, in which the direction of the acting forces and, in many cases, also the linear dimensions and angles are also given. These diagrams are invaluable: they make mistakes arising from ambiguous or cumbrous expression in the wording of the question almost impossible.

On page 221 is given a graphical Time-table for the trains that run between Furstenwalde and Guben by way of Frankfort (on the Oder), a distance of about 80 kilometres. It is well known that these graphical time-tables are largely employed in railway offices for interpolating special trains and for many other purposes. A great deal of information and instruction may be obtained from the inspection of such a time-table.

The book is one which it is a delight to handle and read, as the print is large and the diagrams excellent. We look forward to the second part, which is to treat of the Mechanics of Liquids, with considerable interest.

R. M. MILNE.

Über eine zeitgemässe Umgestaltung des Mathematischen Unterrichts an den höheren Schulen. By F. KLEIN. Pp. 82. 1904. (Teubner.)

This is a set of five addresses delivered at the Easter vacation course for mathematical and physical schoolmasters at Gottingen in 1904, supplemented by reprints of four earlier papers. The author pleads hard for the early introduction of the notation of the calculus into the school curriculum. The principles of the calculus have long ago been admitted into the school course, and it is illogical (to say the least of it) to obtain the critical values of a function by "Schellbach's rule" and to discriminate between maxima and minima by "Fermat's rule," when the methods are those of the calculus pure and simple. If a mathematical pupil becomes gradually familiar with the calculus at an early stage of his career, he will not have to experience a "revolution of ideas" later on. Pupils who are to study law, medicine, chemistry, etc., instead of mathematics at the University need the proposed reform still more urgently. They have no time to continue their mathematical studies at the University, and so under the present system are for ever excluded from the more important mathematical developments of their subject. Their only hope is to read some abridged treatise of the *Calculus for Chemists* type, which they can never really understand, since they have had no suitable training at school.

Other reforms are suggested; artificial problems should be excluded, the graphical illustration of simple functions should be taught early, more stress should be laid on the applications of mathematics (especially to physics), and so on. The practical difficulties involved in the change are given a fair discussion, but this part of the book will naturally be of less interest to English readers. H. H.

Plane Trigonometry. By JAMES M. TAYLOR. Pp. 171. 3s. 6d. 1904. (Boston: The Athenæum Press.)

The chief merit of this book is the large number of examples of the "heights and distances" type. These are first introduced at the very beginning of the book, and are well calculated to stimulate a pupil's interest in the subject. Circular measure is postponed till after the solution of oblique triangles, and is followed by a discussion of graphs, limits, complex numbers, etc., which is too brief to be of any value. "Csc A," "Center," "quality of a line" have a rather curious appearance to English eyes. H. H.

Theoretical Geometry for Beginners. Part IV. By C. H. ALLCOCK. Pp. 224. 1s. 6d. 1904. (Macmillan & Co.)

The first half of this book covers Euclid VI., and is well illustrated by an abundant supply of examples of varying difficulty. The latter half supplies a very interesting introduction to the theory of harmonic ranges, poles and polars with respect to a circle, centres of similitude, inversion, maxima and minima treated geometrically, and envelopes. The theory of ranges other than harmonic would be a welcome addition to future editions; the book is sufficiently attractive to make the reader wish for more. H. H.

Elementary Trigonometry. By C. H. P. MAYO. Pp. 204. 1904. (Longmans, Green & Co.)

This book is divided into a practical and a theoretical section. The former begins with a definition of sine, cosine, and tangent applicable to angles of any magnitude and a table of these functions for acute angles. This table is at once applied to the finding of heights and distances and to the solution of oblique triangles by splitting them up into two right-angled triangles. The second section begins with the definition of cosecant, secant, and cotangent, and ends with circular measure and the obtaining of the trigonometrical functions of 30°, 45°, etc. It is a pity that so little stress is laid on the more usual methods of solving triangles. H. H.

Elementary Plane Geometry. By V. M. TURNBULL. Pp. 136. 2s. (Blackie & Son.)

This excellent text-book is based on the report of the Cambridge Syndicate (1903), and is intended for students who have already been through a preliminary course of experimental work. The proofs given are clear and concise, but rigorous: *e.g.* Euclid I. 5 is proved by folding about the bisector of the vertical angle, but it is first shown that this bisector exists. The whole book is illustrated by a plentiful and well-chosen supply of practical and theoretical exercises. The proofs of the theorems alone are given, the proofs of the constructions being left to the pupil. The ground covered is that of Euclid I. to VI.; the theorems of Euclid II. are rightly considered as illustrating algebraic formulæ, not as proved by those formulæ.

H. H.

Elementary Pure Geometry, with Mensuration. By E. BUDDEN, M.A., B.Sc. (Chambers.) 1904.

Mr. Budden's book is full of interest, but the method of treatment is probably too condensed for due assimilation by a beginner. Moreover, in some cases the proofs are incomplete, thus in obtaining the locus of points (in a plane) equidistant from two given points he proves that all points on the perpendicular bisector satisfy the condition, but does not show that points not on that line fail to satisfy the condition. This same neglect of the second of the two conditions for a complete locus occurs elsewhere, and spoils the logic of proof based on such loci.

Nevertheless, for older students who wish to have all their geometrical mathematics comprised in a single book, and who are able to correct such slips on their own account, the book is likely to be very useful. It is most comprehensive, including the substance of Euclidean Plane and Solid Geometry with extensions to centres of similitude, antiparallels, etc., elementary trigonometry to solution of triangles, harmonic and anharmonic ratios, and other concepts of modern geometry, geometry of conic sections, and mensuration of solids.

The printing is clear, and the collection of examples excellent, and although some of the earlier proofs may be found fault with on the score of rigorousness, the terse vigour of the proofs in general is most remarkable, and in this respect the book will be found invaluable. Mr. Budden's treatment of "ratio" is worthy of minute consideration, as it appears to cover the case of incommensurables with easy naturalness. The book should be in every mathematical library.

A. LODGE.

Mathematical Problem Papers. By E. M. RADFORD. Pp. 203. 1904. (Cambridge University Press.)

This is a collection of one hundred problem papers, each consisting of twelve questions, intended for the use of candidates for entrance scholarships at Cambridge. A large number of the problems are original and the rest have been compiled from various University and College examination papers "other than entrance scholarship papers," so that the work does not enter into competition with the well-known volumes of scholarship papers issued by the University Press.

Indeed what must strike anyone looking through this collection is the unfamiliarity of the great bulk of the questions. Occasionally one comes across a question from the *Educational Times* (lxxxvi., 3), or the *Gazette* (xlix., 7), or Roberts' *Dynamics* (viii., 11), or Besant's *Dynamics* (xlii., 11), or an old Tripos question (xlix., 11). For the most part, however, the old, tried friends of one's youth have quietly dropped out, and their places are filled by strangers of a later generation and of the most uncompromising aspect. Though, of course, there are scattered here and there a few easy questions, the papers, which are graduated in degree of difficulty, proceed from ordinary flagellation to chastisement with scorpions. One might safely predict that any scholarship candidate who had been nurtured upon, and could assimilate, such strong diet as is here provided, would not only be absolutely certain of success, but could also count upon a respectable place in the Tripos. It might be possible to work steadily through the first half of the book (one paper a week) in the year before going up to Cambridge, and to finish the other half during the first year of residence. This, of itself, would almost prove a sufficient problem equipment for the first four days of the Tripos.

The geometrical questions are sometimes peculiar statements of old properties. For instance (xl., 8) "Prove that all parabolas with a given self-conjugate triangle touch four fixed straight lines." The author has evidently followed recent work in the *Gazette* when he gives (xciii., 1) "Prove that the operations of inversion with respect to two circles are commutative only when the circles cut orthogonally." One of H. E. Dudeney's puzzles appears in (xlix., 1) "Given two points in a plane find a compass construction for the mid-point of their join." This little study in stigmatic geometry will well repay attention.

The analytical portions of the work are most distinguished. In (xxiii., 7) the equation is given to the lines joining the origin to the four points of intersection of two conics $u_0+u_1+u_2=0$ and $v_0+v_1+v_2=0$ in the simple form.

$$\begin{vmatrix} u_0 & u_1 \\ v_0 & v_1 \end{vmatrix} \cdot \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = \begin{vmatrix} u_0 & u_2 \\ v_0 & v_2 \end{vmatrix}^2$$

In (xliii., 8) the equation to the four normals from (h, k) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is stated as

$$\{a^2(x-h)^2+b^2y-k)^2\}(kx-hy)^2=(a^2-b^2)^2(x-h)^2(y-k)^2,$$

which is easily obtained by identifying the equations $y-k=\lambda(x-h)$ and $ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$.

A great "tour de force" is displayed in (vi., 7), where the author virtually writes down without calculation the equation to the five sides of a regular pentagon. Choosing suitable origin and axes this equation in polar coordinates is

$$\prod_{s=0}^{s=4}[r\cos(\theta-2s\pi/5)-a]=0,$$

or

$$r^5[\cos 5\theta-\cos(5\cos^{-1}a/r)]=0$$

$$r^5(16\cos^5\theta-20\cos^3\theta+5\cos\theta)-16a^5+20a^3r^2-5ar^4=0,$$

or

$$5(x^2+y^2)^2(a-x)-20(x^2+y^2)(a^3-x^3)+16(a^5-x^5)=0.$$

There are very few misprints or inaccuracies. Question (lxxxviii., 6) should read: "The pedal line of any point P on the circum-circle of the triangle ABC cuts the sides at distances x, y, z from the circum-centre. Prove that $\Sigma x^2 \,.\, \sin 2A$ is constant" (Bell & Abbott's Scholarship).* This is a curious property the geometrical meaning of which is not apparent.

For those who are desirous of following the trend and the scope of present day scholarship papers there could be no better guide than the present collection. The many things hard to be understood will doubtless prove humbling to pride. But if professional dexterity in problem-solving has to be acquired (and when acquired, maintained) what more invigorating training in this direction could be found than that which is afforded by a persistent and resolute struggle with such veritable Hackenschmidts as these? R. F. DAVIS.

Theory of Observations. By T. N. THIELE, Directory of the Copenhagen Observatory. London, C. and E. Layton (56 Farringdon Street). Pp. 144 + vi. 4to. 1903. 12s.

This work on the Theory of Errors and Method of least Squares can be heartily commended. The treatment is not in the least of the narrow, detailed, and arithmetical kind one might expect; the broad and philosophic mind of the author is reflected on every page, and the discussion on the basis of the theory will be found interesting by many who have never been attracted by its crystallisation into practice.

Teachers who are accustomed to include the elements of the theory of probability in their algebra course will find a welcome stimulus in Professor Thiele's book. Most of it can be read in an arm-chair. E. T. WHITTAKER.

A Treatise on Hydromechanics. Part I. **Hydrostatics.** By W. H. BESANT and A. S. RAMSEY. Sixth Edition. Pp. 264. 1904. (Bell.)

Dr. Besant's *Hydrostatics* is too well known to University men to require much more than a passing reference, and congratulations to the author on seeing the sixth edition of this work. But all things change, and it is rare to find anything stereotyped in Mathematics for long. The influence of the doctrine of energetics has been recognised by the author and his collaborator. The chapter on the stability of equilibrium of floating bodies has been enlarged, and the subject is also treated from the point of view of the principle of energy. So, also, the laws of capillarity have been deduced by the same principle from the assumption of the existence of forces between the molecules. Many familiar examples from the sixties and seventies have disappeared from the new edition, and their places have been filled by others set at recent examinations. We may venture to express regret that Dr. Besant has not seized the opportunity of including in the book historical notes, which would be of great interest to the student. The practical value of the book would have also been enhanced had descriptions been given in some detail of the best known instruments, and it certainly would have made the book more attractive to the average student.

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* Math. Trip. 1899 [W. J. G.].

Scientia. Exposé et Développement des questions scientifiques à l'ordre du jour. Published under the direction of MM. Appell, d'Arsonval Haller, Lippmann, Moissan, Poincaré, Marey, etc. 2 frcs. each volume. 1903 – 1904. (Gauthier-Villars.)

This excellent little series, originally published by M. Naud, has now been taken in hand by Messrs. Gauthier-Villars, from whom we have received some half-dozen sample copies. One of these very handy and useful books has already been reviewed in these columns—Laurent's *Elimination*—and Hadamard's *Essay on Taylor's Series* was entrusted by a *too confiding editor to a reviewer* some two years ago, and that reviewer's conscience will, we hope, experience a sharp prick should his eye meet these lines. The volumes are monographs of about 80-100 pages each. *Géométrographie*, or the art of geometrical constructions, is dealt with by that past master in the art, M. Lemoine, whose name is so intimately connected with the somewhat futile but very ingenious researches of some fifteen years into what was called the *Geometry of the Triangle*. To it we hope to recur. *Diagrammes et Surfaces thermo-dynamiques* is a French translation of the first two memoirs written by the late Willard Gibbs *on a subject which is closely connected with the growth of the study of the phenomena of equilibrium in physical and chemical systems. By the application of the dissipation principle,* Gibbs was able to construct his theory of systems maintained at a uniform and constant temperature by the gain or loss of external heat - systems known as non-isolated but isothermal. Of his work Professor Larmor says: "His monumental memoir made a clean sweep of the subject; and workers in the modern experimental science of experimental chemistry have returned to it again and again to find their empirical principles forecasted in the light of pure theory, *and to derive fresh* inspiration for new departures." The memoirs were too long inaccessible to the ordinary student. Ostwald published, in 1891, a German translation of the three classical memoirs which had appeared, in 1873-1878, in the Transactions of the Connecticut Academy, where for long they remained in obscurity. Le Châtelier published a French translation of the essential part of the third memoir in 1899, and this monograph in *Scientia* contains a translation of the first two memoirs. It will familiarise the reader with the ideas of Willard Gibbs on the geometrical representation of thermo-dynamical phenomena by diagrams and surfaces. We hope to have this reviewed more in detail at some future period. The other volumes are not entirely within the purview of the Gazette. M. Cotton writes on the Zeeman Effect. A bright double line is produced if a sodium flame is placed in front of the slit of a spectroscope. But if the flame is placed between the poles of a powerful magnet these lines are found to be broadened and separated into three components. These components are polarised in directions at right angles to each other, the middle line in one plane and the two outer in another. The researches of Lorentz and many others are given in detail by M. Cotton, and he gives an account of Michelson's investigations by the interference method, and of his wonderful echelon spectroscope, which has a much greater resolving power than the grating spectroscope. Zeeman's observation of the effect of magnetism on the rays of light is of great interest to Englishmen, inasmuch as Faraday sought for it in vain "at the time Gauss was in search of the keystone of electro-dynamics" (*v.* that valuable and remarkable book, Merz's *History of European Thought in the Nineteenth Century*, Vol. ii., p. 197. Blackwood.) M. Mendelssohn deals with *Electric Phenomena in Living Beings*; M. Imbert with the *Methods of Economic Functional Activity of the Organism*. The last of the set received is by Le Dantec, and is entitled *La Sexualité*. The author is one of the most daring of modern writers on the mysteries of life and death, and this monograph should be read for its brilliant speculations on some of the most absorbing problems in general biology. We should not under ordinary circumstances venture to draw attention here to books dealing with these subjects. But there must be many among our younger readers who rest under some disadvantage in their desire to become acquainted with the trend of modern thought in the regions of physics and biology; and we think that in this series they will find the pabulum for which they crave, readily accessible, and at a most moderate cost.

Exercises Méthodiques de Calcul Integral. By E. BRAHY. Pp. viii, 302. 1903. (Gauthier-Villars.)

The private student will find this excellent collection of examples of considerable value, if read and worked out *pari passu* with the usual text-book. The first ten chapters take the reader up to volumes and surfaces. The next eight deal with elementary differential equations. A goodly number of the integrals and equations are worked out at length, and to each section and chapter are appended plenty of worked exercises. The fact that new editions of the exercises in the Differential Calculus, by the author of this little volume, and of Frenet's *somewhat similar collection of exercises on the Infinitesimal* Calculus, have appeared within the last couple of years would seem to shew that books of this class appeal to a large number of students abroad. In view of the stress that is to be laid in future in the Inter. Sci. and B.Sc. on the knowledge of French and German, this would form a useful volume for those who wish to make themselves acquainted in the former language with the various technical expressions of the Calculus.

Obras sobre Mathematica. By F. GOMES TEXEIRA. Vol. i., pp. 402. 1904. (The University Press, Coimbra.)

This handsome volume, published by Professor Texeira at the instance of the Portuguese Government, contains the first instalment of the many memoirs and *notes of the author, which until now have been scattered throughout the pages of* mathematical journals in French, Spanish, Portuguese, Italian, and German, the greater number being in the first-named language. They deal with both analytical and geometrical subjects, and include what we believe is the best piece of work the author has done—his researches on the development of functions in series, which *have appeared at intervals during the last seven or eight years.* Other interesting articles treat of curves parallel to the ellipse, the equipotential curve, Descartes' Ovals, Pascal's Limaçon, and the convergence of formulae of interpolation. Many indeed are the claims on the Treasury of this country, but in our wildest dreams we have never thought of asking the Government of the day to publish even the collected works of a Cayley, a Sylvester, or a Stokes.

Scherz und Ernst in der Mathematik. By W. AHRENS. **Geflügelte und Ungeflügelte Wörte.** Pp. x. 522. 1904. (Teubner.)

This entertaining volume consists of extracts, winged and otherwise, from the writings, addresses, letters, etc., of the most famous German, French, and English mathematicians. A few examples will give an idea of the miscellaneous nature of the contents. "Geometry is the only science which has produced no sects" (Frederick the Great). "D'Alembert, that great genius, seems to be far too ready to pull down everything he has not himself built up" (Euler to Lagrange). "I am delighted at the contrast between your modesty and the *good opinion that other geometers have of themselves, although they have* certainly nothing like the same claim. You are a living instance of what you said to me some time ago, that pretensions are ever in an inverse ratio to merit" (d'Alembert to Lagrange). "Leibnitz never married. At the age of 50 he began to think about it, but the lady asked for time to reflect. This gave Leibnitz time to reflect—and he did not marry" (Fontenelle). "My dear and illustrious friend—they write to me from Berlin that you are about to take what we philosophers call 'le saut perilleux,' and that you have married one of your relations. . . . Accept my compliments, for a mathematician ought to have pre-eminent advantages in the calculations of—his own happiness, and any calculations of yours are sure to lead to a solution—the solution in your case being marriage" (d'Alembert to Lagrange).

"Petit Poisson deviendra grand
Pourvu que Dieu lui prête vie" (Laplace of S. D. Poisson).

"Sophie Germain's conversation had all the elegance of one of Laplace's beautiful formulae" (Fayolle). The determination of the present Ministry in France to have none but true republicans in the Army reminds us that history repeats itself. It is interesting to recall that the taking of the oath to Napoleon III. was dispensed with in the case of only two men—Arago and Cauchy. Arago had views on the subject, as the following dialogue will shew:

The Minister (*X.*)*: Now, sir! are you really one of us?*

Fresnel: No one is more devoted than I am to our august Royal House, and to the learned institution which France owes to that House.

X.: Yes! but *that is too vague.* Suppose you were a deputy, which section of the deputies would you join?

Fresnel: I would join Camille Jordan, if he would have me.

X.: I am much obliged for your frankness.

(Next day an unknown gets the post for which Fresnel was applying.)

"My friend! let us take a pinch of snuff; to-morrow I shall not enjoy that pleasure, for my hands will be fastened behind my back" (Bailly to his nephew, the night before his execution). And so on every page. We cannot refrain from quoting in conclusion Hermite's epitaph on Cauchy:

Venerandae memoriae
Eximii viri Augustini Cauchy
Hoc qualecunque munus
Libenter accipias,
Et, ex beata aeternae felicitatis sede
Animum sanctae amicitiae pie memorem
Benigne aspicere digneris.

Lehrbuch der Differential und Integral-Rechnung. By J. A. SERRET. Edited by G. BOHLMANN and E. ZERMELO. Vol. II. Conclusion. Pp. xii, 305-479. 1904. (Teubner.)

This is a translation of Chapters XI. and XII. of Serret's well-known book. It contains an appendix by Axel Harnack on the integration of partial differential equations in the theory of functions of a complex variable.

Elementare Algebra. Akademische Vorlesungen für Studierende der ersten Semester. By E. NETTO. Pp. viii. 200. 1904. (Teubner.)

This would be found an excellent little manual for the scholarship candidate who requires revision of the algebraical part of his course. It deals, and on general lines, with equations of the first, second, third, fourth, and *n*th degrees, the binomial and polynomial theorems, combinations, and determinants with their applications to linear equations with more than one unknown. Incidentally there is a good deal of advanced work in these chapters. For example, in the chapter on the general equation of the second degree there are questions on the probability of real roots, or that the roots are of the same sign. Graphical illustrations appear where required.

Practical Geometry for Beginners. By V. LE NEVE FOSTER and F. W. DOBBS. Pp. viii, 96. 1904. (Macmillan.)

This is a good book and has its points. The subject matter is that of the first book of Euclid, "for it seems wiser to avoid trespassing on later principles, and thereby spoiling the student by a premature acquaintance of the joys of subsequent discoveries." The second part is intended to be used concurrently with a course of theory, cross references being appended for those who are reading Mr. Allcock's *Theoretical Geometry for Beginners.*

There are plenty of miscellaneous examples, and patterns are introduced in order to stimulate the pupil to invent designs.

NOTE.

In a notice of Professor Bellino Carrara's "Trisezione dell' Angolo" (Vol. iii., p. 65) the writer states that there is no mention in this *brochure* of Klein's well-known *Vorträge über ausgewählte Fragen der Elementargeometrie.* He now begs to make the *amende honorable* to Professor Carrara. It is mentioned on p. 17.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) For Sale.

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882 Vols. I.-IX. Edited and published by E. Hendricks, M.A., Des Moines, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by Artemas Martin, M.A. (Erie, Pa.)]

The Mathematical Monthly. Vols I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. Runkle, A.M.

Proceedings of the London Mathematical Society. First series, complete Vols. 1-35. Bound in 27 vols. Half calf. £25

Cayley's Mathematical Works Complete, equal to new, £10. Apply. Professor of Mathematics, University College, Bangor.

(2) Wanted.

The Messenger of Mathematics. Vols. 24, 25.

Tortolini's Annali. Vol. I. (1850), or any of the first eight parts of the volume.

Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

BOOKS, ETC., RECEIVED.

On the General Equations of Mechanics. By P. E. B. Jourdain. Pp. 61 79. Quart. Jour. Math. No. 141 1904.

The Oxford and Cambridge Graphical Algebra By R. H. Allpress and F. Marshall. Pp. 26. 6d. 1904. (Gill.)

Woher kommen die Weltgesetze By J. Kleiber. Pp 30 1 m 1904. (Teubner.)

Elementary Trigonometry. By C. H. P. Mayo Pp. x, 204. 3s. 6d. 1904 (Longmans.)

An Introduction to the Modern Theory of Equations. By Florian Cajori. Pp. x, 233. 7s. 6d. net. 1904. (Macmillan.)

Encyclopédie des Sciences Mathématiques Pures et Appliquées. Tome I. Vol. 1. Fascicule I. Edited by Professor Jules Molk and F. Meyer. French Edition. Pp. 160. 5 fr. or 4 mk. 1904. (Gauthier Villars or Teubner.)

American Journal of Mathematics. Edited by Frank Morley. Vol. XXVI No. 4. Oct., 1904. (Kegan Paul.)

Invariants of Linear Partial Differential Equations and the Theory of Congruences of Rays. E. J. Wilczynski. *On Elements connected each to each by one or Two Reciprocal Relations.* C. de Polignac.

The Lobachevski Prize. By G. B. Halsted. (*Science*, Sept. 16.)

Über eine zeitgemässe Umgestaltung des mathematischen Unterrichts an der hoheren Schulen. By F. Klein Pp. 82 1 m. 60. 1904. (Teubner.)

Beiträge zur Frage des Unterrichts in Physik und Astronomie an den hoheren Schulen. By E. Reicke. Pp. 190. 2 m. 1904 (Teubner.)

Methodisches Lehrbuch der Elementarmathematik By G. Holzmüller. Vol. I. Pp. 317. 2 m. 80 1904. (Teubner.)

Scherz und Ernst in Mathematik By W. Ahrens. Pp. 522. 8 m. 1904. (Teubner.)

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THE
MATHEMATICAL GAZETTE.

EDITED BY
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THE DEVELOPMENT OF GEOMETRICAL METHODS.

V.

FROM this moment a brilliant period opens for geometrical research of every kind. Analysts interpret all their results, *and set to work to translate them by constructions*. Geometers endeavour to discover in every question some general principle—in most cases impossible to prove without the aid of Analysis—so as to deduce a host of particular consequences, closely connected with each other and with the principle from which they are derived. Jacobi's brilliant pupil, Otto Hesse, admirably develops the methods of homogeneous coordinates of which Plucker, perhaps, did not fully appreciate the value. Boole discovers in Bobillier's polars the first notion of a covariant, the theory of forms is created by the labours of Cayley, Sylvester, Hermite, and Brioschi. Aronhold, Clebsch, Gordan, and other geometers still with us, invent the definitive notation of the theory, establish the fundamental theorem relative *to the limitation of the number of covariant forms*, and thus succeed in giving to the theory its fullest extension and scope.

The theory of surfaces of the second order, constructed mainly by the school of Monge, is enriched by a large number of elegant properties—mostly due to Hesse, who later finds in Paul Serret a worthy rival and an investigator who will continue his work.

The properties of the polars of algebraical curves are developed by Plucker, and in particular, by Steiner. The study of curves of the third order, which have by this been the subject of research for a considerable time, is enriched and rejuvenated by a host of new ideas. In the first place, Steiner treats by pure geometry the double tangents of curves of the fourth order, and Hesse, following in his steps, applies algebraical methods to this beautiful problem, as well as to the points of inflexion of curves of the third order.

The idea of *class*, introduced by Gergonne, the study of a paradox partially elucidated by Poncelet, relating to the respective degrees of two curves each the polar reciprocal of the other, are the immediate impulse to Plücker's researches on the so-called *ordinary* singularities of algebraical plane curves. The celebrated formulæ discovered by Plücker are later extended by Cayley and by other geometers to algebraical gauche curves, and by Cayley again and by Salmon to algebraical surfaces. Singularities of a higher order are in their turn attacked by geometers; Halphen shews that, contrary to an opinion which then widely obtained, each of these singularities may be considered as equivalent to a certain group of ordinary singularities, and his researches bring to a close for the time being this difficult and important question.

Analysis and Geometry—Steiner, Cayley, Salmon, Cremona—meet in their investigations into surfaces of the third order; and as Steiner foresaw, this theory becomes as simple and as easy as that of surfaces of the second order.

Ruled algebraical surfaces, so important in their applications, are studied by Chasles, by Cayley, the marks of whose influence are seen in every form of mathematical investigation, by Cremona, Salmon, and La Gournerie; and later by Plücker in a volume to which I shall presently recur.

The study of the general surface of the fourth order still appears to be too difficult; but that of the particular surfaces of that order with multiple points or multiple lines is begun by Plücker for wave surfaces, by Steiner, Kummer, Cayley, Moutard, Laguerre, Cremona, and many others. As for the theory of algebraical gauche curves, its elements are extended, and finally it receives through the investigations of Halphen and Noether the most notable advancement. Between their labours it is impossible on such an occasion as this to draw a distinction. A new theory with a great future before it is initiated by Chasles, Clebsch, and Cremona; it deals with all the algebraical curves which can be traced on a given surface.

Homography and correlation, the two methods of transformation which originated all the preceding investigations, receive in their turn an unexpected development; they are not the only methods which bring a single element into correspondence with a single element, as is shewn by a particular transformation briefly pointed out by Poncelet in his *Traité des propriétés projectives.* Plücker invents *transformation by reciprocal radii vectores* or *inversion,* and it was not long before Sir William Thomson and Liouville shewed its importance both in Mathematical

Physics and in Geometry. Magnus, a contemporary of Möbius and Plücker, maintained that he had found the most general transformation which will bring a point into correspondence with a point. But Cremona's researches shew that Magnus transformation is only the first of a series of birational transformations which the great Italian geometer shews us how to obtain methodically, at any rate for the figures of Plane Geometry. Cremona's transformations will retain their interest for a long time to come, although later researches proved that they always reduce to a series of successive applications of the transformation we owe to Magnus.

VI.

All the investigations I have mentioned, and others to which I shall presently recur, originate and find their first impulse in the conceptions of modern geometry. But I must now point out another stimulus to great progress in geometrical research. Legendre's Theory of Elliptic Functions, which had been too much neglected by French geometers, was developed and extended by Abel and Jacobi. With these great geometers, almost immediately followed by Weierstrass and Riemann, the theory of Abelian functions, which Algebra will presently attack by its own unaided resources, will bring to the Geometry of curves and surfaces a contribution the importance of which will continue to increase.

Jacobi had already used the analysis of elliptic functions to prove Poncelet's celebrated theorems on in- and circum-scribed polygons, thus opening a chapter which has been enriched by a large number of elegant results; by methods connected with geometry, he had also succeeded in integrating Abelian equations

But Clebsch was the first to shew in a long series of investigations the whole importance of the notion of the *deficiency* of a curve, due to Abel and Riemann, by developing a vast number of results and elegant solutions, which the use of Abelian integrals appeared, so simple was it, to connect with their real point of departure. The study of the points of inflection of curves of the third order, that of double tangents of curves of the fourth order, and, in general, the theory of osculation on which the ancients and moderns had so often exercised their minds, were connected with the beautiful problem of the division of the elliptic and Abelian functions.

In one of his memoirs, Clebsch had discussed *rational* curves, or curves of deficiency zero; this led him, towards the end of a life alas! too short, to consider what may be called *rational* surfaces, those which can be simply represented by a plane. Here was a vast field for research, already opened for the elementary cases by Chasles, in which Clebsch was followed by

Cremona and many other savants. In this connection Cremona, generalising his researches in plane geometry, brought to light not the whole of the birational transformations of space, but some of the most interesting of these transformations The extension of the idea of deficiency to algebraical surfaces has already begun; already also work of great value has shewn that the theory of simple or multiple integrals of algebraical differentials will find, in the study of curves and surfaces alike, an extensive field of important applications, but it is not for me to dwell any further on this topic

VII

While the mixed methods of which we have just pointed out the principal applications were being created, pure geometers did not remain inactive Poinsot, the creator of the theory of couples, developed by a purely geometrical method, described by him as "one in which the object of research is never for one moment out of sight," the theory of rotation of a solid body which the investigations of d'Alembert, Euler, and Lagrange seemed to have exhausted, Chasles made a valuable contribution to kinematics by his beautiful theorems on the theory of the displacement of a solid body, which have since been extended by other elegant methods to the case in which the motion has various degrees of freedom He published his beautiful theorems in the theory of attraction which are worthy to rank with those of Green and Gauss Chasles and Steiner found themselves on common ground in the study of the attraction of ellipsoids, and thus shewed once more that geometry has its place marked out for it in the most important questions of the integral calculus.

Steiner did not disdain to devote his attention at the same time to the elementary portion of geometry His researches on the contacts of circles and conics, on isoperimetrical problems, on parallel surfaces, on the centre of gravity of curvature,* aroused the admiration of all by their simplicity and their depth

Chasles introduced his principle of correspondence between two variable objects, which has given birth to so many applications; but here analysis assumed its rights, and studied the essentials of the principle, giving it precision, and generalising it This was also the case with the famous theory of *characteristics* and the numerous researches of Jonquières, Chasles, Cremona, and others, who were to lay the foundation of a new branch of geometry—*Enumerative Geometry*. For several years the validity of Chasles' celebrated postulate was unchallenged; many geometers thought that it had been irrefutably established. But, as Zeuthen observed at the time, it is very difficult in proofs of this kind to recognise that there is always some weak point which the author may not have perceived; in fact, Halphen, after many fruitless attempts, definitively crowned all these researches by clearly indicating in what cases Chasles' postulate may be accepted, and when it may be rejected.

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* *Krummungsschwerpunkt* [Tr.]

VIII.

These are the chief investigations which restored synthetic geometry to its place of honour, and assured to it in the course of the last century its due place in mathematical research. Many illustrious workers have taken part in this great geometrical movement, but it must be recognised that their leaders and guides were Chasles and Steiner. Such was the splendour of the light cast by their wonderful discoveries that, at any rate for the moment, they threw into the shade the publications of other modest geometers, who were perhaps less desirous of discovering brilliant applications likely to kindle a love for geometry, than of building the science itself upon an absolutely solid foundation Perhaps the labours of the latter received a more tardy reward, but their influence increases daily, and no doubt will increase still more. There is no doubt that were I to pass them by in silence, I would be neglecting one of the principal factors which will play their part in future researches. It is especially to von Staudt that I am here alluding. His geometrical work has been expounded in two works of great interest · the *Geometrie der Lage*, which appeared in 1847, and the *Beitrage zur Geometrie der Lage*, published in 1856, that is to say, four years after the *Géométrie Supérieure*

Chasles, as we have seen, was attempting to build up a body of doctrine independent of the Cartesian analysis, and had not completely succeeded I have already pointed out one of the reproaches that may be levelled at this system. imaginary elements are only defined in it by their symmetrical functions, which necessarily excludes them from a considerable number of investigations On the other hand, the constant use of anharmonic ratio, transversals, and involution, requiring frequent analytical transformations, gives to his *Géométrie Supérieure* an almost exclusively metrical character, which notably differentiates it from the methods of Poncelet. Returning to these methods, von Staudt endeavoured to construct a geometry free from all metrical relations, and exclusively based upon relations of situation. It is in this spirit that his first work, the *Geometrie der Lage* (1847) was conceived. The author takes as his point of departure the harmonic properties of the complete quadrilateral and those of homologous triangles, proved solely by considerations of geometry of three dimensions, analogous to those of which the school of Monge has made so frequent a use.

In the first part of his work, von Staudt entirely omitted imaginary relations. It is only in his *Beiträge*, his second work, that, by a very original extension of Chasles's method, he geometrically defined an isolated, imaginary element and distinguished it from its conjugate. This extension, although rigorous, is laborious and very abstract. It may be in substance defined as follows: two conjugate imaginary points may always be considered as the double points of an involution on a real line; so that we pass from an imaginary to its conjugate by changing i into $-i$, and as we may distinguish the two imaginary points by making correspond to each of them one of the two different directions that may be attributed to the line. There is something rather artificial in this; the development of the theory erected on such foundations is necessarily complicated. By purely projective methods, von Staudt established a complete method for calculating the anharmonic ratios of the most general imaginary elements. As in the case of all geometry, projective geometry employs the notion of order, and order involves number. It is not therefore surprising that von Staudt was obliged to build up in detail his method of calculation, but the ingenuity which he displayed in arriving at his conclusions must be admired. In spite of the efforts of the distinguished geometers who have attempted to simplify its exposition, we fear that this part of von Staudt's geometry, like the otherwise so interesting geometry of that profound thinker Grassmann, will not prevail against the analytical methods which have now come into almost universal favour. Life is short; geometers know and also practise the principle of least action. In spite of these fears, which should discourage no one, it seems to me that, in the first form that was given to it by von Staudt, projective geometry must necessarily become the companion of descriptive geometry; that the one is called upon to revive the other in spirit, processes, and applications. This has already been felt in several countries, and notably in Italy, where the great geometer Cremona did not disdain to write for schools an elementary treatise on projective geometry.

IX.

In the preceding sections I have tried to follow and to exhibit clearly the remotest results of the methods of Monge and Poncelet. By inventing tangential and homogeneous coordinates, Plücker had seemed to exhaust all that the method of projections and of polar reciprocals could supply to analysis. Towards the end of his life he returned to his early researches, and gave them an extension which was to widen in unexpected directions the domain of geometry.

Preceded by innumerable researches on systems of straight lines by Poinsot, Möbius, Chasles, Dupin, Malus, Hamilton, Kummer, Transon, and especially by Cayley, who was the first to introduce the notion of line coordinates,—researches which originated either in statics and kinematics, or in geometrical optics,—Plücker's line geometry will always be regarded as the part of his work in which we meet with the newest and most interesting of his ideas. It is important that Plücker should have been the first to build up a methodical treatment of the straight line as an element, but that is nothing compared to what he discovered. It is sometimes said that the principle of duality brings into evidence the fact that the plane as well as the point may be considered as an element of space. That is true, but by adding to the plane and the point the straight line as a possible element of space, Plücker was led to recognise that any curve or surface may also be considered as elements of space, and thus sprang into existence a new Geometry, which has already inspired a large number of treatises, and which will stimulate even more in the future. A beautiful discovery of which I shall speak further on has already connected the geometry of spheres and that of straight lines, and has suggested the introduction of the notion of the coordinates of a sphere. The theory of systems of circles has already commenced. No doubt it will be developed with the detailed investigation of the representation, which we owe to Laguerre, of an imaginary point in space by an orientated circle. But before describing the development of these new ideas which have given fresh life to the infinitesimal methods of Monge, I must retrace my steps for a moment, to touch on the history of the branches of geometry which I have so far neglected.

X.

Among the works due to the school of Monge, I limit myself here to the consideration of those which are connected with *finite* geometry; but some of his pupils devoted themselves in particular to the development of the new ideas of infinitesimal geometry, which were utilised by their master in the study of curves of double curvature, lines of curvature, and the generation of surfaces—ideas which are partially expounded in the *Application de l'Analyse à la Géométrie*. Among them we may mention Lancret, the author of beautiful researches on gauche curves, and especially Charles Dupin, the only one, perhaps, who followed all the paths which were opened up by Monge.

Among other works, we owe to Dupin two volumes, the authorship of which Monge would not have been ashamed to acknowledge: the *Développements de Géométrie pure*, which appeared in 1853, and the *Applications de Géométrie et de Mécanique* which dated from 1822. In them we find the idea of the *indicatrix* which was, according to Euler and Meunier, to revolutionise the whole theory of curvature, the conceptions of conjugate tangents and of asymptotic lines which have taken so important a place in recent researches. We cannot forget the determination of the surface, the lines of curvature of which are circles; nor, in particular, the Memoir on triply orthogonal surfaces, in which is to be found, together with the discovery of the triple system formed of surfaces of the second degree, the celebrated theorem with which the name of Dupin will always be connected.

Under the influence of these works, and of the revival of synthetic methods, the geometry of the infinitely small resumed in all researches the permanent position which Lagrange had hoped to mark out for it. It is a remarkable thing that the geometrical methods thus revived were to receive their liveliest impulse on the publication of a memoir which at first, at least, seemed to be connected with analysis pure and simple. I mean Gauss's celebrated volume *Disquisitiones generales circa superficies curvas*, which was published in 1827 at Göttingen, and the appearance of which may be said to mark an epoch in the history of infinitesimal geometry.

From this moment the infinitesimal method received such an impulse in France as it had not yet experienced. Frenet, Bertrand, Molins, J. A. Serret, Bouquet, Puiseux, Ossian Bonnet, and Paul Serret developed the theory of gauche curves. Liouville, Chasles and Minding combined with them in methodically following up Gauss's memoir. Jacobi's integration of the differential equation of the geodesic lines of the ellipsoid inspired many investigations. At the same time the problems studied in Monge's *Applications de l'Analyse* were largely developed. The determination of all surfaces having their lines of curvature plane or spherical completed in the most happy manner some of the partial results already obtained by Monge. It was now that one whom Jacobi considered the most penetrating of geometers, Gabriel Lamé (who like Charles Sturm had begun with pure geometry, and had already made the most interesting contributions to that science in a little work published in 1817, and in Memoirs which appeared in the *Annales de Gergonne*), utilised the results obtained by Dupin and Binet on the system of confocal surfaces of the second degree. He, conceiving the idea of curvilinear co-ordinates of space, created an entirely new theory which was destined to receive applications of the most varied kind in mathematical physics.

(*To be continued.*)

ON 'CIRCULAR MEASURE' AND THE PRODUCT FORMS OF THE SINE AND COSINE.

I. CIRCULAR MEASURE.

WERE it not for Euclid's Propositions iii. 26 (proved by superposition) and vi. 33, the statement that he nowhere defines the circumference of a circle or any curved line would surely be a truism. His definition of a circle is substantially only a locus definition. It may well be argued that neither does he define a straight line (or an angle); but in the case of a straight line he introduces sufficient ideas and axioms to make his subsequent propositions convincing, whereas he gives us nothing to bridge the interval between a straight line and a curved one.

And so, wise after the event, we may say that all the thought spent in old days over 'squaring the circle' had no adequate foundation, and was foredoomed to fail.

As to iii. 26 and vi. 33, they are only further examples of the truth with which we are familiar in the case of angles, that it is sometimes easier to prove that things are equal etc., than to say exactly what they are.

The following is the result of an attempt to reach by the simplest means from established straight line results a satisfactory idea of an 'arc of a circle.' The use of the Trigonometrical Functions does not indicate a lack of simplicity, these, as used, being nothing but convenient words for expressing simple straight line ratios.

The following propositions are established in what follows:

I. $n \sin \frac{A}{n}$ increases and $n \tan \frac{A}{n}$ diminishes as n increases, A representing any angle with no reference to its 'measure.'

II. Hence the combined lengths of 'inscribed chords' and of 'circumscribed tangents,' as they increase in number, approach one another (and therefore some fixed quantity) indefinitely. This quantity may be called the length of the arc or the circular measure of the angle subtended.

III. This quantity is proportional to the angle it subtends at 'the centre.'

I. Let Figs. 1 and 2 (p. 130) represent n angles at a point O, each being x, ($nx <$ a right angle), and A, 1, 2, ... n points at unit distance from O along their arms. In Fig. 1 straight lines are drawn from 1, 2, ... n, parallel to OA, and in Fig. 2, $O1$, $O2$ ... On are produced, to meet in both cases the straight line from A at right angles to OA.

Then by Definition, An' is the *sine* of nx, and An'' is the *tangent* of nx.

$$\therefore\ \frac{\sin nx}{n} \text{ is the average of } A1',\ 1'2', \ldots$$

and
$$\frac{\tan nx}{n} \text{ is the average of } A1'',\ 1''2'', \ldots.$$

The lines $A1$, 12, $23 \ldots$ are of course all equal.

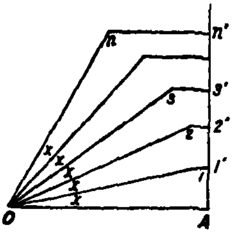

Fig. 1.

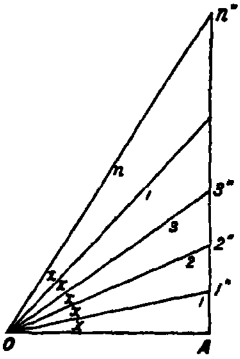

Fig. 2.

Now in Fig. 1 these equal straight lines $A1$, $12 \ldots$ make constantly increasing angles with An' and $\therefore$ the intercepts $A1'$, $1'2'$, etc., constantly diminish.

$\therefore$ their average diminishes as n increases.

$$\text{i.e. } \frac{1}{n}\sin nx \text{ diminishes as } n \text{ increases.}$$

But in Fig. 2, the equal lines being at a constant angle with the rays from O, the lines $A1''$, $1''2''$, $2''3'' \ldots$ constantly increase.

$\therefore$ their average increases as n increases.

$$\text{i.e. } \frac{1}{n}\tan nx \text{ increases with } n.$$

Writing

$\frac{A}{n'}$ for nx, and therefore $n'\frac{x}{A}$ for $\frac{1}{n}$, n' diminishes as n increases;

$\therefore$, provided $\frac{A}{n'} <$ a rt. $\angle$,

$$n'\sin\frac{A}{n'} \text{ increases, and } n'\tan\frac{A}{n'} \text{ diminishes, as } n' \text{ increases.}$$

II. Now $\sin\frac{A}{n} = \tan\frac{A}{n} \cdot \cos\frac{A}{n}$, and $\cos\frac{A}{n} < 1$.

$$\therefore\ n\sin\frac{A}{n} \text{ is always less than } n\tan\frac{A}{n}.$$

$$\therefore\ n\sin\frac{A}{n} \text{ and } n\tan\frac{A}{n} \text{ approach one another.}$$

Further, their difference is

$$n\tan\frac{A}{n}\left(1-\cos\frac{A}{n}\right)=n\tan\frac{A}{n}\,.\,2\sin^2\frac{A}{n}=2n\tan^3\frac{A}{n}\,.\,\cos^2\frac{A}{n},$$

which $\quad < 2n\tan^3\frac{A}{n}$ or $\frac{2}{n^2}\left(n\tan\frac{A}{n}\right)^3$.

Now when n is large, $n\tan\frac{A}{n} < p\tan\frac{A}{p}$, where p is anything less than n; and for all finite values of A, p can be chosen so that $p\tan\frac{A}{p}$ is finite.

$$\therefore\ \text{the difference} < \frac{2}{n^2}\times(\text{a finite quantity}),$$

and $\therefore$ diminishes indefinitely as n increases.

$\therefore$ $n\sin\frac{A}{n}$ and $n\tan\frac{A}{n}$ approach one another continuously, and $\therefore$ both approach some fixed quantity a, which is *defined by this continuous approach of $n\sin\frac{A}{n}$ and $n\tan\frac{A}{n}$.*

But in Fig. 1 (with all restrictions as to the size of the angles removed, except that $\frac{A}{2n} <$ a rt. $\angle$) the sum of $A1$, $12\ldots$ is $n\,.\,2\sin\frac{A}{2n}$ if the whole angle is called A, and, if lines are drawn at A, 1, 2 ... at rt. $\angle$s to OA, $O1$, $O2\ldots$ to meet, the sum of these is $2n\tan\frac{A}{2n}$.

$\therefore$ a lies between the sums of these 'inscribed chords' and 'circumscribed tangents.'

The limit of either of these sums of straight lines is what we mean by the 'arc of the unit circle' from A to n, or the 'circular measure' of the angle A.

III. To prove that the quantity a (*i.e.* the length of the 'arc') is proportional to A.

Since $\quad a < \left|\begin{matrix} n\tan\frac{A}{n} \\[6pt] n\sin\frac{A}{n}\end{matrix}\right|$ and $\beta < \left|\begin{matrix} n'\tan\frac{B}{n'} \\[6pt] n'\sin\frac{B}{n'}\end{matrix}\right|$, however great n and n' may be,

$$\frac{\alpha}{\beta} < \frac{n}{n'} \left|\begin{array}{c|c} \dfrac{\sin\frac{A}{n}}{\sin\frac{B}{n'}} & \begin{array}{c}\sec\frac{A}{n}\\[1ex] \cos\frac{B}{n'}\end{array}\end{array}\right|.$$

Now assume n and n' to be so chosen that

$$\frac{A}{n}=\frac{B}{n'}; \text{ then } \frac{n}{n'}=\frac{A}{B}$$

(the actual magnitude of n and n' remaining free);

$$\therefore \quad \frac{\alpha}{\beta} < \frac{A}{B} \left|\begin{array}{c}\sec\frac{A}{n}\\[1ex] \cos\frac{A}{n}\end{array}\right|, \text{ however great } n \text{ may be.}$$

Now the difference

$$\sec\frac{A}{n}-\cos\frac{A}{n}=\frac{\sin^2\frac{A}{n}}{\cos\frac{A}{n}}=\tan^2\frac{A}{n}\,.\,\cos\frac{A}{n}=\frac{1}{n^2}\left(n\tan\frac{A}{n}\right)^2\cos\frac{A}{n}$$

which is seen, as above, to diminish indefinitely as n increases.

And $\therefore$ the only quantity which can be said to lie between

$$\frac{A}{B} \left|\begin{array}{c}\sec\frac{A}{n}\\[1ex] \cos\frac{A}{n}\end{array}\right|,$$

when n is indefinitely increased, is $\frac{A}{B}$.

$$\therefore \quad \frac{\alpha}{\beta}=\frac{A}{B} \text{ (for all values of } A \text{ and } B\text{).}$$

This proves that α as defined in § II. above is a sound 'measure' of the angle A.

[NOTE 1.—Any difficulty that may seem to arise from incommensurable angles is covered by the *meaning* of $\frac{A}{B}$ in such a case. We take it to mean 'something which lies between $\frac{A'}{B}$ and $\frac{A''}{B}$, where A', A'' are angles which are commensurable with B,' one being greater and the other less than A, but both indefinitely near to A, $\frac{\alpha}{\beta}$ having a similar meaning, we cannot escape the conclusion that with these definitions $\frac{a}{\beta}=\frac{A}{B}$, as for commensurables.]

[NOTE 2.—The 'circular measure' of an angle is perhaps better named 'the arc' of the angle. All measurements of an angle are circular, the length of the arc being compared in the rectangular systems with the length of the circumference, and in 'circular measure' with the length of the radius. The use of 'number of radians' as a definition seems unnatural and unsatisfactory. The *radian* is an uninteresting angle—Lord Kelvin introduced the word merely as a convenience in lecturing, to avoid the long phrase 'angle whose circular measure is.']

II. THE PRODUCT FORMS OF THE SINE AND COSINE.

If n is a positive integer, the addition formulae give $\cos nx$ as an algebraic function of degree n in $\cos x$ only.

Hence

$$\cos 2nx-\cos 2na$$
$$\equiv C.^{*}(\cos x-\cos x_1)(\cos x-\cos x_2)\ldots(\cos x-\cos x_n),$$

where $\cos x_1$, $\cos x_2$, ..., represent the roots (not yet proved real) of

$$\cos 2nx=\cos 2na.$$

Now this equation is satisfied for all values of x for which $2nx=4i+2na$, the angles being measured in right angles and i representing any integer. Thus $x=\frac{2i}{n}+a$.

But the $2n$ values of x given by $i=0, 1, 2, \ldots, 2n-1$ have all different cosines, as the sum of any two angles is $\frac{2i+2j}{n}+2a$, and the difference is $\frac{2i-2j}{n}$ and none of these is a multiple of 4, since $i-j<2n$;

$\therefore$ the $2n$ values of $\cos\left(\frac{2i}{n}+a\right)$ are all roots of $\cos 2nx=\cos 2na$,

and $\therefore$ $\cos 2nx-\cos 2na\equiv C.\prod_{0}^{2n-1}\left[\cos x-\cos\left(\frac{2i}{n}+a\right)\right]$.

Now to obtain $\sin nx$, put $a=0$ in this identity.

Then $\cos 2nx-1$, and therefore

$$\sin^2 nx\equiv C.\prod_{0}^{2n-1}\left[\cos x-\cos\frac{2i}{n}\right].$$

Now $\cos\frac{2i}{n}=\cos\frac{2j}{n}$ if $2i+2j=4n$, *i.e.* $j=2n-i$;

---

* C. is used throughout merely for the word 'constant': its value changes from line to line.

∴ the pairs of values of i, $\frac{1}{2n-1}, \frac{2}{2n-2}, \ldots, \frac{n-1}{n+1}$, give equal cosines. $i=0$ and $i=n$ are peculiar terms, and

$$\sin^2 nx \equiv \text{C.}\,(\cos x-1)(\cos x+1)\,.\prod_1^{n-1}\left(\cos x-\cos\frac{2i}{n}\right)^2;$$

$$\therefore\ \sin nx \equiv \text{C.}\,\sin x\,.\prod_1^{n-1}\left(\cos x-\cos\frac{2i}{n}\right).$$

Again $\cos\frac{2i}{n} = -\cos\frac{2j}{n}$ if $2i+2j=2n$; ∴ $j=n-i$;

∴ the pairs of values of i, $\frac{1}{n-1}, \frac{2}{n-2}, \ldots$, give opposite cosines, and $i=\frac{n}{2}$ is a peculiar term if n is even;

∴ $\sin nx \equiv \text{C.}\,\sin x\,.\prod_2^{n-1\text{ or }2}\left(\cos^2 x-\cos^2\frac{\text{even}}{n}\right)$, with an extra factor $\cos x$ if n is even.

But $\cos^2 x-\cos^2\frac{\text{even}}{n} = \sin^2\frac{\text{even}}{n}-\sin^2 x$, and also can be put in the form

$$\cos^2 x\sin^2\frac{\text{even}}{n}-\sin^2 x\,.\cos^2\frac{\text{even}}{n}$$

or

$$\cos^2 x\,.\cos^2\frac{\text{even}}{n}\left(\tan^2\frac{\text{even}}{n}-\tan^2 x\right);$$

$$\therefore\ \sin nx \equiv \text{C.}\,\sin x\,.\prod_2^{n-1\text{ or }2}\left(1-\frac{\sin^2 x}{\sin^2\frac{\text{even}}{n}}\right),\text{ with }\cos x\text{ if }n\text{ is even,}$$

and also $$\equiv \text{C.}\,\tan x\,.\prod_2^{n-1\text{ or }2}\left(1-\frac{\tan^2 x}{\tan^2\frac{\text{even}}{n}}\right).\cos^n x.$$

By making x very small we get $\text{C.} = \frac{\sin nx}{\sin x}$ or $\frac{\tan nx}{\tan x}$, both of which are n.

Now write $\frac{x}{n}$ for x.

Then

$$\sin x \equiv n\sin\frac{x}{n}\,.\prod_2^{n-1\text{ or }2}\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{\text{even}}{n}}\right),\text{ with }\cos\frac{x}{n}\text{ if }n\text{ is even,}$$

and $$\equiv n\tan\frac{x}{n}\,.\prod\left(1-\frac{\tan^2\frac{x}{n}}{\tan^2\frac{\text{even}}{n}}\right).$$

Now, if ξ is the circular measure of x, that is $\frac{x\pi}{2}$, $\sin\frac{x}{n} < \frac{\xi}{n}$ and $\tan\frac{x}{n} > \frac{\xi}{n}$.

Moreover, if $x < 2$, by § II. above,

$$\frac{\sin\frac{x}{n}}{\sin\frac{\text{even}}{n}} > \frac{x}{\text{even}} \quad\text{and}\quad \frac{\tan\frac{x}{n}}{\tan\frac{\text{even}}{n}} < \frac{x}{\text{even}};$$

$$\therefore\ \sin x < \xi \cdot \prod_{2}^{n-1 \text{ or } 2}\left(1 - \frac{x^2}{\text{even}^2}\right)$$

and

$$> \xi \cdot \prod_{2}^{n-1 \text{ or } 2}\left(1 - \frac{x^2}{\text{even}^2}\right) \cdot \left(\cos\frac{x}{n}\right)^n;$$

$$\therefore\ \sin x < \xi\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{4^2}\right)\dots\left(1 - \frac{x^2}{\text{even}^2}\right)\frac{1}{\left(\cos\frac{x}{n}\right)^n},$$

where n is either of the two numbers succeeding the last 'even' of the product.

Now $\left(\cos\frac{x}{n}\right)^n > \left(1 - \frac{\xi^2}{2n^2}\right)^n$, which $> 1 - \frac{\xi^2}{2n}$;

$$\therefore\ \text{finally,}\quad \sin x = \frac{\pi}{2} \cdot x\left(1 - \frac{x^2}{2^2}\right)\left(1 - \frac{x^2}{4^2}\right)\dots\left(1 - \frac{x^2}{\text{even}^2}\right) \cdot \left[1 - \theta\frac{\xi^2}{2n}\right],$$

where θ is a proper fraction;

$\therefore$ the greater n is, the nearer will $\sin x$ be to the product, and the error gets smaller and smaller without limit.

We now proceed to show that the limitation $x < 2$ above can be removed.

If $x > 2$, the factors

$$\left(1 - \frac{\sin^2\frac{x}{n}}{\sin^2\frac{2}{n}}\right), \quad \left(1 - \frac{\tan^2\frac{x}{n}}{\tan^2\frac{2}{n}}\right)$$

are not respectively less and greater than $\left(1 - \frac{x^2}{2^2}\right)$; but $\left(\text{assuming } n \text{ so great that } \frac{x}{n} < 1\right)$ both these factors are, for $x > 2$, negative, and since now, by § II. above,

$$\frac{\sin\frac{x}{n}}{\sin\frac{2}{n}} < \frac{x}{2} \quad\text{and}\quad \frac{\tan\frac{x}{n}}{\tan\frac{2}{n}} > \frac{x}{2},$$

the corresponding positive factors

$$\left(\frac{\sin^2\frac{x}{n}}{\sin^2\frac{2}{n}}-1\right),\quad\left(\frac{\tan^2\frac{x}{n}}{\tan^2\frac{2}{n}}-1\right)$$

are less and greater than $\left(\frac{x^2}{2^2}-1\right)$, and so for $x>4$, 6, ...

The final result is therefore true for all values of x.

For the cosine,

put $na=1$ in (I.), then $\cos 2na=\cos 2=-1$;

$$\therefore\ \cos 2nx+1\equiv C.\prod_{0}^{2n-1}\left[\cos x-\cos\frac{2i+1}{n}\right].$$

Now $\cos\frac{2i+1}{n}=\cos\frac{2j+1}{n}$ if $2i+2j+2=4n$; $\therefore\ j=2n-i-1$,

so that the pairs $\frac{0}{2n-1}, \frac{1}{2n-2}, \dots \frac{n-1}{n}$, give equal cosines.

$$\therefore\ 2\cos^2 nx\equiv C.\prod_{0}^{n-1}\left[\cos x-\cos\frac{2i+1}{n}\right]^2;$$

$$\therefore\ \cos nx\equiv C.\prod_{0}^{n-1}\left(\cos x-\cos\frac{2i+1}{n}\right).$$

Now $\cos\frac{2i+1}{n}=-\cos\frac{2j+1}{n}$ if $2i+2j+2=2n$, *i.e.* $j=n-1-i$,

so that the pairs $\frac{0}{n-1}, \frac{1}{n-2}, \dots,$

give opposite cosines, and $i=\frac{n-1}{2}$ is peculiar if n is odd;

$$\therefore\ \cos nx\equiv C.\prod_{1}^{n-1\text{ or }2}\left(\cos^2x-\cos^2\frac{\text{odd}}{n}\right),\ \text{with } \cos x \text{ if } n \text{ is odd},$$

$$=C.\,\Pi\left(1-\frac{\sin^2 x}{\sin^2\frac{\text{odd}}{n}}\right)=C.\,\Pi\left(1-\frac{\tan^2 x}{\tan^2\frac{\text{odd}}{n}}\right).\cos^n x\ \text{(as before)},$$

with an extra factor $\cos x$ in the former if n is odd.

$$\therefore\ \cos x=C.\prod_{1}^{n-1\text{ or }2}\left(1-\frac{\sin^2\frac{x}{n}}{\sin^2\frac{\text{odd}}{n}}\right)=C.\prod_{1}^{n-1\text{ or }2}\left(1-\frac{\tan^2\frac{x}{n}}{\tan^2\frac{\text{odd}}{n}}\right)\cos^n\frac{x}{n}.$$

Putting $x=0$, we see $C=1$;

$$\therefore \text{ as before, } \cos x < \Pi\left(1-\frac{x^2}{\text{odd}^2}\right)\left|\begin{matrix}1\\ \cos^n\frac{x}{n}\end{matrix}\right|,$$

where n is either of the two numbers succeeding the last 'odd' in the product;

$$\therefore \text{ as before, } \cos x = \left(1-\frac{x^2}{1^2}\right)\left(1-\frac{x^2}{3^2}\right)\ldots\left(1-\frac{x^2}{\text{odd}^2}\right)\cdot\left[1-\theta\frac{\xi^2}{2n}\right],$$

x being rectangular measure, and ξ its circular measure.

Corollary. By comparing these product forms with the expansions, we have

$$(1)\quad \xi.\Pi\left(1-\frac{x^2}{\text{even}^2}\right)\cdot\left[1-\theta\frac{\xi^2}{2n}\right] \equiv \Sigma\left(\pm\frac{\xi^{\text{odd}}}{\underline{\lvert\text{odd}}}\right)+\frac{\xi^n}{\underline{\lvert n}}\left|\begin{matrix}\sin x\\ 0\end{matrix}\right|;$$

$$(2)\quad \Pi\left(1-\frac{x^2}{\text{odd}^2}\right)\cdot\left[1-\theta\frac{\xi^2}{2n}\right] \equiv \Sigma\left(\pm\frac{\xi^{\text{even}}}{\underline{\lvert\text{even}}}\right)+\frac{\xi^n}{\underline{\lvert n}}\left|\begin{matrix}\sin x\\ 0\end{matrix}\right|.$$

From which follow without difficulty

$$\sum_2^{n-1\text{ or }2}\frac{1}{\text{even}^2}+\frac{\theta}{2n}\cdot\frac{\pi^2}{4}=\frac{\pi^2}{4\,\underline{\lvert 3}};$$

$$\therefore \sum_2^{n-1\text{ or }2}\frac{1}{\text{even}^2}=\frac{\pi^2}{24}\left|\begin{matrix}1\\ 1-\frac{1}{12n}\end{matrix}\right|$$

and

$$\sum_1^{n-1\text{ or }2}\frac{1}{\text{odd}^2}=\frac{\pi^2}{8}\left|\begin{matrix}1\\ 1-\frac{1}{n}\end{matrix}\right|.$$

W. N. ROSEVEARE.

MATHEMATICAL NOTE.

162. [X. 5.] *Extract from an old pamphlet on the slide rule.*

The authors introduced a modification of the old Gunter's rule, and remark in their preface, "And what further recommends it to the world is the *Commodiousness* thereof and the *Manner of Use*; for in *other* instruments you are obliged to state the question right to find the answer, but in this if you state the question *improperly* it will come *right* notwithstanding which renders it *advantageous* and *easy* to those that are unskilled in *Arithmetic* or *common slides.*"

The pamphlet is intituled: The Art of | Measuring made Easy | By the Help of | a new Sliding-Rule | which | performs the same at one operation | as requires two three or more | on the sliding rules heretofore used | In a Plain Easy and Concise Method | entirely New | By W. Bradford R. Darby and J. Hulls Philomaths all of Campden in Gloucestershire. Worcester. Printed by H. Berrow, in Goose Lane 1754.

'Is there not an opening at the present day for an instrument which will furnish the right answer to the wrong question? C. S. J.

REVIEWS.

Quadratic Partitions. By Lieut.-Col. ALLAN CUNNINGHAM, R.E. Pp. xxiii and 266. 12s. 1904. (Francis Hodgson.)

This book mainly consists of tables of the solutions of

$$t^2 + Du^2 = p,$$

where D is any positive or negative number less than 20, and p is a prime number; t and u are the "unknowns" which are given in the tables. For given values of D, and p, the equation may be impossible, and in that case the tables state the fact. If the equation is possible, and D is positive, the Theory of Numbers shews that the solution is unique. But if D is negative and the equation is possible, there is an infinite number of solutions, all of which can be derived from any one of them. In this case the author gives the least solution. The principal table gives the solutions for the important cases $D = 1$, 2, and 3, for all values of p up to 100,000. The only previous tables for these values of D are now difficult to obtain, and are of much smaller extent than the present table, being complete only up to about 13,000. The introduction gives an interesting account of some of the uses in the Theory of Numbers to which the tables can be put.

No one can adequately conceive the labour involved in the calculation of these tables, unless he has himself tried to solve such equations. For some values of D, Gauss, Jacobi, and others have found explicit solutions.

For instance, if $p = t^2 + u^2$, where t is odd, and $p = 4n + 1$, then $t \equiv \frac{1}{2} \frac{2n!}{n!\,n!}$, mod. p; this is theoretically important, but (like most direct methods in the Theory of Numbers) quite impracticable for the purpose of calculating t. Consequently the solutions for each p have to be found individually by a process of trial. We would suggest to the author the desirability of putting his methods on record in any future edition or extension of these tables. We are glad to see it stated in the Introduction that the author (who in 1900 published a "Binary Canon," shewing residues of powers of 2, mod. p) has in preparation other tables relating to the Theory of Numbers.

A. E. WESTERN.

Leçons de Trigonométrie rectiligne. By C. BOURLET. Pp. 322. 1904. 6 francs. (Paris, Colin & Cie.)

This work is one of the series of elementary mathematical text-books edited by M. Darboux. Its three parts include the properties of triangles and quadrilaterals with their associated circles; there is also an appendix on the graphical representation of complex quantities, De. Moivre's theorem, the trigonometrical functions of submultiple angles, and the solution of binomial, quadratic, and cubic equations.

The book may with advantage be put into the hands of a clever schoolboy who has acquired a considerable knowledge of the subject from more elementary text-books. It is clear, suggestive, and very complete; for instance, three different ways of solving $a \cos x + b \sin x = c$ are discussed at length. Examples for the student are scarce; there are only 105, including those in the appendix. The author begins with definitions of vectors and projections, and develops his subject on the lines suggested in the *Mathematical Gazette* of October, 1904, pp. 82 to 85. Particularly interesting are the chapters on the solution of trigonometrical equations, the conversion of a sum into a form suitable for logarithmic calculation, and the solution of triangles. The printing and engraving are beyond praise. H. H.

A New Trigonometry for Schools. By W. G. BORCHARDT and A. D. PERROTT. Pp. 400+xii. 1904. 4s. 6d. (G. Bell & Sons.)

This is a rather more conservative treatise than the majority of recent trigonometries, but it contains numerous concessions to modern theories of teaching. The new and old do not appear as yet to agree well together; but nevertheless the book is distinctly good and should be especially useful to boys who intend to make a serious study of mathematics. The explanations are very full and the examples numerous; few pupils will want to read the whole of the book. The expansions for $\sin\theta$ and $\cos\theta$, and a very clear description of the slide rule, bring the treatise to a close. H. H.

Carl Gustav Jacob Jacobi. By LEO KOENIGSBERGER. Pp. xviii.+554. 1904. (Teubner.)

The hundredth anniversary of Jacobi's birth has been celebrated by the appearance of a biography closely following the lines of the author's *Hermann von Helmholtz*, and amply maintaining the high standard of his former work. Throughout the book it is Jacobi as a mathematician who interests the author rather than Jacobi as a man. A short account of each of Jacobi's numerous contributions to science is given in chronological order, and a very welcome summary of his whole work closes a volume which is as attractive as author and publisher can make it.

Jacobi was the son of Jewish parents and, though known at school as an unusually clever boy, did not reveal exceptional mathematical ability till towards the close of his undergraduate days at the University of Berlin. His inaugural dissertation at the end of his University course first brought him prominently forward as a mathematician, and secured for him in 1826 the post of Privatdozent at Königsberg. It was about this time that he formed the friendships with Bessel and others which so profoundly affected his future labours. From the day he went to Königsberg to the day of his death he worked unceasingly at his contributions to Mathematical Science, and his name rapidly became famous throughout Europe. He was promoted in turn to the posts of Professor Extraordinarius and Professor Ordinarius at Konigsberg, and in 1844 was made Fellow of the Berlin Academy, that he might recruit his failing health and devote himself still more exclusively to research. He died of smallpox in 1851 after a brief illness.

Jacobi's labours covered a wide range of subjects. In addition to his work in the theory of transcendental functions, differential equations, calculus of variations, and analytical mechanics, with which his name will always be most closely associated, he made valuable contributions to Astronomy, the theory of series, etc. In everything he endeavoured, not so much to solve isolated problems, however attractive, as to find the intimate connexion underlying the whole of mathematical truth. H. H.

A Handy Book of Logarithms, with Practical Geometry Appendix. Pp. 128. 2s. 1904. (Blackie.)

This book consists of tables of logarithms of numbers from 1 to 10,000; hyperbolic logarithms of numbers from 1·01 to 30; circumferences, diameters, and areas of circles; square roots and cubes; lengths of circular arcs, given their degrees or given their heights; areas of circular segments up to a semi-circle; trigonometrical ratios from 0° to 90°, advancing by the sixth of a degree; logarithmic sines, cosines, etc.; the points of the compass, and the reciprocals of numbers from 1 to 1000. The first two tables are to four places of decimals; of the rest some are to three places, some to five, and some to six. The appendix contains some of the simple problems in practical geometry, rules on mensuration, etc. The tables are very clearly printed.

Plane Geometry, Practical and Theoretical. By J. S. MACKAY. Books I., II., and III. **Congruent and Equivalent Figures,** corresponding to Euc. I.-IV. Pp. viii, 237. 2s. 6d. (Chambers.)

A short introduction familiarises beginners with the use of the protractor, etc. Dr. Mackay dryly remarks in his preface that before going among foreigners it is well to know something of the language the foreigner speaks. Hence the old plan of placing the definitions first is followed by the author. Dr. Mackay has skilfully adapted the corresponding portion of his edition of the Elements to the requirements of modern programmes of geometrical teaching. He has utilised the excellent collection of exercises with which that edition was adorned, and we also find the historical notes which were so interesting and useful a feature. In addition, this little volume is distinguished from all its competitors by the fact that it is the first to contain a sketch of Geometrography. Dr. Mackay is a careful teacher, and will be found a safe guide for the young student. Beginners will not be much disturbed by the faultiness of the logic in the treatment of tangents to circles by the method of limits. We note that Book I. (128 pp.) is published separately at a shilling.

Encyclopédie des Sciences Mathématiques Pures et Appliquées. French edition, published under the direction of Jules Molk. Tome I., Vol. I. Fascicule I. **Arithmétique.**

We heartily welcome the appearance of the French edition of this monumental Encyclopaedia. Not only does it place within the reach of those who are not accomplished German scholars a most valuable storehouse of information of a kind not elsewhere attainable in so convenient and precise a form, but it is in itself evidence of that *entente cordiale* which reigns throughout the scientific world. For the authors of the original articles in the German edition have collaborated with their French *confrères*, have intimated where, on wider consideration, or for the purposes of a French audience, modifications were advisable or necessary; while, on the other hand, the Frenchmen, on their part, have carefully discussed the proposed changes, and, in particular, have added considerably to the bibliographical information, as may be seen by a glance at the footnotes in asterisks which with the bottom of every pages is sown.

The peculiar value of such a work as this at the present time is obvious. Into every department of science the specialist has penetrated; but the departments of science are not cut and dried divisions. No specialist can afford to be ignorant of what is going on even in fields of research that are apparently as wide as the poles asunder as far as his own investigations are concerned. Again and again have the difficulties in the path of an investigator been removed by an appeal to the researches of others in apparently unconnected regions of research. Who would suppose in the early days of Legendre that the theory of elliptic functions would ever play a part in the development of geometry? And do we not see at the present moment how the theory of groups is making its influence felt in the most remarkable manner in almost every department of mathematics? It is clear that the world of mathematical thought must lie under a deep debt of obligation to Prof. Klein and his band of collaborators, who have made it possible for either the normal mathematician or the specialist in a particular field to refer to what has been done by other workers in the developments of the theories in which they have been interested.

In the fascicule which lies before us we have three sections. First we have 63 pages on the fundamental principles of Arithmetic, in which the article originally due to H. Schubert is expounded by Messrs. Tannery and Molk. The idea of the natural number, from Leibnitz to Frege, Peano, and Russell; numeration; direct and inverse operations; operations of every kind, from the simplest, such as additions to Mannoury's hypermultiplication—all in turn are concisely dealt with, and furnished with copious lists for further reference.

The second section treats of the Combinatory Analysis and the Theory of Determinants—Netto's article, expounded by H. Vogt. The final section is not concluded in this part. It deals with irrational numbers and the idea of the limit, and is interpreted by Prof. Molk for the French reader from the German of A. Pringsheim. Here it is sad to notice that although there are plentiful references to English writers in the second section, from Wallis to MacMahon, yet in section iii. the only English name revealed by a cursory glance is that of Newton!

It is hardly necessary to add that we feel assured that as soon as the quality of the work in this encyclopaedia is known to the younger generation of mathematicians, they will not be happy till they get it. Klein and his colleagues have indeed conferred upon us a boon which will be generously recognised.

Technical Mechanics. By E. R. MAURER. Pp. xvii, 403. $4. 2nd edition, revised and enlarged, 1904. (New York: Wiley & Sons; London: Chapman, Hall.)

We are glad to see that the merits of Professor Maurer's treatise on Technical Mechanics have so rapidly been recognised. We have already commented in favourable terms on the excellent planning of the broad outlines of this book, and on the skill and judgment with which the illustrative examples of practical interest have been selected and arranged (vol. iii., p. 16). The additions to the present edition consist of an appendix on the principle of virtual work as applied in the case of a particle, a system of particles, and to statically indeterminate problems; an appendix containing sections on truss loads, weight of roofing, weight of trusses, weight of snow, wind pressure, computation of apex loads, maximum stresses, classification of frames, and additional remarks on funicular polygons.

Solutions of the Problems and Theorems in Charles Smith's Geometrical Conics. By CHARLES SMITH. Pp. 143. 6s. 1904. (Macmillan.)

Key to Godfrey and Siddons' Geometry. By E. A. PRICE. Pp. 172. 5s. 1904. (Cambridge University Press.)

Solutions of the Examples in Hall's Graphical Algebra. By H. S. HALL and H. C. BEAVEN. Pp. 59. 3s. 6d. 1904. (Macmillan.)

Key to Elementary Geometry. By CECIL HAWKINS. Pp. 195. 5s. 1904. (Blackie.)

There will be many teachers to whom Mr. Smith's key will be a boon, and many private students who will find it of the greatest value if they have no other guidance. We are glad to see that the price is prohibitive to the ordinary school-boy, although when a boy is old enough to tackle the stiffer examples in Mr. Smith's *Conics* he is quite old enough to know how to use a key wisely and well. We have tested a good few of the solutions and found them neat and elegant, as indeed one would expect from a teacher of Mr. Smith's experience. We may add that the solutions refer to the fourth edition of the *Geometrical Conics*.

We may compliment Mr. Price on the figures in his *Key to Godfrey and Siddons' Geometry*, and, in fact, we ought to extend the compliment to the Cambridge University Press for their share in its production. There must be a very large number of teachers to whom this little book will be of value, partly because practical geometry was not in their mathematical course while at the University and partly because the figures and solutions will save them much time in examining the works of their classes. The figures are reduced copies of diagrams drawn to scale, and in most cases Mr. Price has given the answers to one place of decimals more than is asked for. Here again the price is probably beyond the reach of all beginners but those drawn from the ranks of the plutocracy.

As one might expect, for the present there will be many to whom graphical work is both novel and difficult. Mr. Hall tells us that he has had many letters of enquiry on the subject of "graphics" from both teachers and students. This has naturally led him to prepare a Key with full solutions and diagrams of all the more important examples. If, as he suggests, the Key is only used as a final check on the result, the appearance of this little volume is amply justified. The 75 graphs are very carefully and accurately drawn. We know from personal experience the time and attention that is required in the correction of the beginners' graphs, and we can cordially recommend this Key to all teachers of large classes using the *Graphical Algebra*. Mr. Hawkins gives no diagrams and, as he says, has found it necessary to give fuller treatment to particular cases than is advisable or necessary in ordinary cases. This will not make his book any the less useful to the teacher, for whom the solutions are intended. Ample hints are given for the solutions of many of the questions, but we rather think that the enterprising school-boy of tender years who purchases this Key for the purpose of diminishing his mental exertions will report to his envious neighbours that he has been "jolly well sold." On the other hand, the solutions are ample for the young teacher or for the more advanced private student who requires some external assistance.

A Short Introduction to Graphical Algebra. By H. S. Hall. Pp. 49. 1s. 1903. (Macmillan.) Second Edition, revised and enlarged.

The Oxford and Cambridge Graphical Algebra. By R. H. Allpress and F. Marshall. Pp. 26. 6d. n.d. (Gill.)

An Elementary Treatise on Graphs. By Professor G. A. Gibson. Pp. x, 183. 3s. 6d. 1904. (Macmillan.)

Graphs and Imaginaries. By J. G. Hamilton and F. Kettle. Pp. 41. 1904. (Arnold.)

The *Short Introduction to Graphical Algebra*, already a new issue, is a considerable improvement on the first edition, mainly owing to the inclusion of applications to practical questions preparing the student for his future work in the laboratory and the workshop. It is now an adequate introduction.

Messrs. Allpress and Marshall's little pamphlet is confined to the consideration of the graphs of simple functions, easy maxima and minima, limiting values, and the use of co-ordinates in finding the areas of triangles. It is well done within these narrow limits, but contains no practical examples.

In the case of Professor Gibson's little treatise we come to another type of book. We wish we had space to quote his wise remarks on the place that graphical methods are taking in the mathematical teaching of the day. The general treatment of the subject is excellent, and a large number of the useful examples are taken from laboratory note-books. Cubic and biquadratic, logarithmic, exponential, and trigonometric functions receive due attention. There are sections on component and resultant graphs, on methods of trial and error, on adiabatic curves and harmonic curves, and the decomposition of a curve into harmonic components. The eighth chapter is a very brief discussion of the equations of the conic sections. The simpler and more important properties are embodied in the selection of examples appended to the chapter, so that the student will be familiar with them even should he not be able to study them more intimately. A few useful tables bring to a close a treatise which is comprehensive and thorough, and which by its simplicity of exposition is convincing evidence of Professor Gibson's skill as a teacher.

One hardly knows what to say about *Graphs and Imaginaries*, if the authors intended it for the use of boys. Assuming that it is intended for children of a larger growth, it may be described as an interesting and ingenious piece of work. The sub-title describes its scope—an easy method of finding graphically imaginary roots of quadratic equations and imaginary points of intersections of various curves, with illustrations of the principle from elementary geometry. To solve the ordinary quadratic the parabola $y^2=ax^2+bx+c$ is drawn on squared paper. We are then shown how to draw what the authors call the *shadow* parabola of this curve, which is easily effected, being only the original turned into another position. The unreal roots of the quadratic lie on this *shadow*. The next chapter exhibits the "circle" method of solving a quadratic and the manner in which the shadow may be obtained. From this the imaginary points of intersection of a line and a circle or of two circles readily follow, and the authors then discuss the tangent to a circle from a point within it, and the imaginary points of contact. The relation between the circle and its *shadow* is next treated, and a few problems are worked out, such as:—Divide a line 4 inches long into two parts such that the sum of the squares on them will be 4 square inches; given a circle radius a, find how far distant from the centre is a chord $2c$ units in length, when c is less and when it is greater than a.

Examples in Algebra selected from Elementary Algebra. By W. M. Baker and A. A. Bourne. Pp. xi, 222, lxxvi. 3s. Part I., without Answers, 1s. 6d.; Part II., without Answers, 2s. 1904. (Bell.)

The boy or girl who has worked through the whole of the examples given by Messrs. Baker and Bourne should be perfectly familiar with the manipulation of algebraical expressions, and quite at home with the ordinary processes. A word of commendation must be added as to the careful graduation of the examples. Graphical methods receive due attention, and there is an ample supply of revision papers.

Arithmetical Examples. By W. G. Borchardt. Pp. 247. 3s. 1904. (Rivingtons.)

Exercises in Arithmetic (Oral and Written). Part III. By C. R. Taylor. Pp. viii, 132, xx. 1s. 6d. 1904. (E. Arnold.)

The second edition of Mr. Borchardt's Arithmetical Examples has been enlarged by the addition of many examples of an "easy problem" type, and also of a large number of easy oral questions. They seem to be well graduated, and no doubt will make their way where teachers are beginning to adopt or exhibit a preference for books of examples without an accompanying text-book.

The second book is the third part of the Exercises compiled from the work done by the author in the Wimbledon High School, and follows closely the lines of the Report of the *Mathematical Association*. They seem carefully selected and graduated. Books such as this and that of Mr. Borchardt represent a considerable amount of patient care and thought, and unless a reviewer has the opportunity of working through much of them in class, he has to be content with expressing a somewhat perfunctory opinion as to their real merits.

Preliminary Practical Mathematics. By S. G. Starling and F. C. Clarke. Pp. 168. 1s. 6d. 1904. (E. Arnold.)

This book is devised for the use of students at technical schools, and seems to be perfectly adapted for the purpose for which it is designed. It contains the irreducible minimum of mathematical knowledge requisite for the intelligent following of the work done in the laboratory and the workshop. It is very depressing to think that such books should be required at all. The same story comes to us from all parts of the country that in these classes, instead of the teacher being able at once to embark on his subject, he has in a large number of cases to teach the would-be student of chemistry or engineering what he ought to have learned at school. As long as this state of affairs persists, we suppose such books as this will be necessary.

Elements of Plane Trigonometry. By R. Lachlan and W. C. Fletcher. Pp. viii, 164. 2s. 1904. (E. Arnold.)

A good deal of skill has been shown in the inclusion within the cover of so small a book of the elements as far as Demoivre's theorem, summation of series, expansions, and factorisations. The readers are assumed to be thoroughly acquainted with the trigonometrical ratios and their applications, so that the introductory pages are to be regarded as an "extension and systematisation of work already done." In the same way, the student is supposed to be acquainted with the use of logarithms. We may draw special attention to the note on Interpolation, p. 12; to the worked out examples on solution of triangles, pp. 76-83, with the hints and observations on the opposite side of the page to the solutions. This book is a bold move in the direction of simplicity, and what the authors have done they have well done.

A School Geometry. Parts I.-VI. **(Containing Plane and Solid Geometry, treated both theoretically and graphically.)** By H. S. Hall and F. H. Stevens. Pp. xiv, 442, ix. 4s. 6d.

A School Geometry. Part VI. **(Containing the substance of Euc. XI., 1-21, together with theorems relating to the surfaces and volumes of the simpler solid figures.)** Pp. 347-442, iv. 1s. 6d. 1904. (Macmillan.)

We have now the whole course of Plane Geometry as set forth by these well-known authors of text-books on elementary mathematics. We notice that enough is shown the student on pp. 270-273 to enable him to proceed at once to the study of such a book as that of Messrs. Lachlan and Fletcher. Let us say that the book is excellent, although it may run on closer lines to Euclid's Elements than is the present fashion in most of the text-books which load our book-shelves. In the eyes of many this will not detract from its merits. The last volume contains figures of the usual nets, with instructions for the making of models therefrom. The "get-up" of the whole series of volumes—paper, printing, and diagrams—is worthy of the firm of Macmillan.

Elementary Geometry of the Straight Line, Circle, and Plane Rectilineal Figures. By Cecil Hawkins. Part II. Pp. 167-296. 2s. 1904. (Blackie.)

We are afraid that one fatal drawback will be found to the general utility of Mr. Hawkins' admirable edition of the Elements of Geometry. It contains a capital collection of well-devised examples, but no numerical answers are given. With this exception, we can only speak of the book in terms of high praise. We presume that the course he has brought so far forward will be completed by a volume dealing with solid geometry, and in a second edition he might be persuaded to follow the example of others and include a simple treatment of the elementary trigonometrical ratios, etc.

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) For Sale.

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I.-IX. Edited and published by E. Hendricks, M.A., Des Moines, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by Artemas Martin, M.A. (Erie, Pa.)]

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. Runkle, A.M.

Proceedings of the London Mathematical Society. First series, complete Vols. 1-35. Bound in 27 vols. Half calf. £25.

Cayley's Mathematical Works. Complete, equal to new, £10. Apply, Professor of Mathematics, University College, Bangor.

(2) Wanted.

The Messenger of Mathematics. Vols. 24, 25.

Tortolini's Annali. Vol I. (1850), or any of the first eight parts of the volume.

Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

BOOKS, ETC., RECEIVED.

A School Geometry. Parts I.-VI. By H. S. Hall and F. H. Stevens. Pp. xiv, 442, xii. 4s. 6d. 1904. (Macmillan.)

On a Method of Dealing with the Intersections of Plane Curves. By F. S. Macaulay. Pp. 385-410. Reprinted by Trans. of Am. Math. Soc. Vol. 5. No. 4. Oct. 1904.

Elementary Algebra. Part II. By W. B. Baker and A. A. Bourne. Pp. vi, 277-468, liii-lxxvi. 1904.

A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, with an Introduction to the Problem of the Three Bodies. By E. T. Whittaker. Pp. xiii, 414. 12s. 6d. 1904. (Cam. Univ. Press.)

Culegere de Probleme de Algebra. (Theory of Equations.) Compiled by A. G. Ioachimescu. Pp. 82. 2.50 lei. 1904. (Göbl, Bucharest.)

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THE
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THE MATHEMATICAL ASSOCIATION.

THE Annual Meeting of the Mathematical Association was held in the Medical Library of King's College, London, on Saturday, 28th January, 1905. An exhibition of apparatus for use in the teaching of mathematics was held in the adjoining room by MR. W. J. BROOKS, of 33 Fitzroy Street, W.; MR. CUSSONS, of Broughton, Manchester; and the Educational Supply Association. About sixty members and visitors were present.

The PRESIDENT (PROFESSOR A. R. FORSYTH) took the chair at 3 o'clock, and called upon one of the Honorary Secretaries to read the Report of the Council.

MR. C. PENDLEBURY read the report as follows:

The Council have to report that no General Meeting was held in May or October, no papers having been offered.

Since the last Annual Meeting 30 new members have been elected, 6 have died, and 18 have retired, resulting in a small net gain of 6 members.

The Association now consists of 402 members, of whom 10 are honorary, 19 life members by composition, 47 life members under the rule which released those who had paid 10 annual subscriptions from all further liability, 30 who were formerly life members under this rule, but who voluntarily resumed payment of the annual subscription, and 296 ordinary members. Briefly, of the 402 members, 19 are life members by composition, and 326 are liable for the payment of the annual subscription.

The six members who have died during the year are:

Professor G. J. Allman, F.R.S., formerly Professor of Mathematics at Queen's College, Belfast.

Dr. R. Biggs, Head Master of the Royal School, Portora, Enniskillen, who was drowned on June 23rd, 1904.

R. W. H. T. Hudson, Fellow of St. John's College, Cambridge, and Lecturer in Mathematics at Liverpool University, who was killed on 20th September, 1904.

The Venerable Archdeacon Hughes-Games, Vicar of Hull, who was an original member of the A.I.G.T.

W. Rheam, of Woodford College, S. Woodford, who died on September 11, 1904.

J. S. Yeo, late Fellow of St. John's College, Cambridge, and Master at Fettes College.

The Council desire to offer their warmest sympathy in particular to Professor W. H. H. Hudson, one of our Vice-Presidents, the name of whose son appears in the above list. Ronald Hudson, in the few years that were allotted to him, had won a wide reputation by his original work, and had won too the personal affection of many friends by his singularly beautiful and loveable character.

In addition to the usual 6 numbers of the *Gazette*, the Council has published two special reports, one on the teaching of elementary mathematics, and one on advanced school mathematics. The editor of the *Gazette* wishes to express his thanks to many coadjutors, and to ask for papers on more elementary matters than he has been able to obtain hitherto.

Professor Forsyth has now served the customary two years of office, and the Council desire to nominate Mr. G. B. Mathews as his successor.

The retiring members of the Council are Mr. Dyer of Eton and Mr. Roseveare of Harrow. The Council nominate as their successors Mr. Dobbs of Eton and Mr. Hawkins of Haileybury.

The President then called upon Mr. F. W. Hill, the Honorary Treasurer, to present his report for the past year. A copy of the accounts accompanies this *Gazette*.

Mr. A. W. Siddons made a brief report of the work of the Committee on the Teaching of Elementary Mathematics, and proposed that the Committee should be re-appointed, which was agreed to.

The reports having been adopted, the nomination of Mr. G. B. Mathews for the office of President for the years 1905-7, and of Mr. Cecil Hawkins and Mr. F. W. Dobbs as members of the Council, were approved.

Mr. G. B. Mathews then took the chair, and a vote of thanks to the retiring President, Professor Forsyth, was passed unanimously on the motion of Dr. F. S. Macaulay, seconded by Mr. C. S. Jackson.

Mr. E. M. Langley then read a paper on "Models and their Uses." This was illustrated by a large number of models, and was followed and afterwards discussed with much interest. The paper will appear in a later number of the *Gazette*.

The second paper was by **Mr. W. H. Wagstaff** on "The New Geometry."[1]

The introduction of "The New Geometry" has brought into prominence many views both of what ought to be taught and how it ought to be taught; the present is a time of unsettlement, and it is only with time, patience, and temperate discussion that we can hope to arrive at a general agreement as to what is best.

We must, as practical men engaged in school teaching, try to frame not merely an ideal course for the individual boy, but the best workable course for the education of boys generally; it follows that we must take account of various disturbing factors. Of these I now mention two—that examinations are, and probably always will be with us, and that boys are not generally educated entirely at one school; these two facts have a very important bearing on the vexed question, shall we, or shall we not, have a fixed order of teaching Geometry?

If each teaching establishment were entirely unconnected with all others and with the rest of the world, each would have its own order, and no difficulty would arise; as things are, however, difficulties may be expected, and my object is to consider not whether a fixed order is best *in itself*, but whether it is best under the circumstances.

First then:—How is the question affected by the fact that candidates from many schools enter for the same public examinations?

In order to make it possible for all candidates to take such examinations, no matter what order of Geometrical study they had followed, some might advocate the total exclusion of bookwork; the paper would then consist entirely of problems, and the student would avail himself of any data with which he was acquainted. This would get over the difficulty; and as we know, there has been a marked tendency in this direction already, but it is doubtful whether all examining bodies would ever adopt it, and still more doubtful whether it is wise; the crammer would regard the proofs of the Propositions as unworthy of attention; he would teach them the facts and not how the facts were obtained, and this would be a great loss, however much the student might be practised in riders.

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[1] This paper has been partly re-written, in consequence of some passages in the original having been misunderstood.

If then we dismiss the idea of confining all Geometrical examinations to problems, by what other method can we avoid the 'order' difficulty? what is to be the criterion as to whether a particular result may or may not be assumed in seeking to prove some other result?

It is most interesting to note how the Cambridge Syndicate meet this. They say, "Any proof of a Proposition shall be accepted which appears to the examiners to form part of a systematic treatment of the subject." Nothing more comprehensive could be desired; but may we not fear that it is too comprehensive? would it not be often impossible for the Examiners to tell whether a proof really deserved to be passed or not? I hope Professor Forsyth will give us the benefit of his opinion about this.

A second objection to the lack of a universal sequence has been already mentioned, viz., that boys frequently begin their Geometry at one school, and continue it at another. The undesirability of changing the order of propositions in the early stages of a boy's geometrical training is so obvious that I feel I need not labour the point, and I content myself with calling attention to it.

For these reasons I think we ought to have a definite order, if only we could agree upon it; and I hope that this Association will take the initiative in establishing such an order.

Leaving this question I pass on. Perhaps I shall be expected to say something on the merits and demerits of the New Geometry itself; but so much has been said already that I cannot hope to say anything new, and hesitate to say anything at all.

Teachers seem to like it, and so do boys; of course this is an important, but not necessarily the most important consideration. Is the mental training sacrificed? I think it need not be, if the subject is properly taught, and rigorous demonstration insisted on, but I believe it often is sacrificed by slurring over difficulties. When I say a rigorous demonstration, I mean one that will satisfy a good but a mere mathematician, as distinguished from a philosopher.

The experimental side of the work should not be overdone. We want, no doubt, to get the finger-training and the observation-training as well as the familiarity with geometrical figures which the experimental work gives, but, after all, these for most of us are but bye-products, however important; the real object of the New Geometry is to make Geometry not less, but more, effective as a training for the reason.

The New Geometry would seem to admit more boys than the old into the realm of reason-training by geometrical methods; it lowers the margin of cultivation; by its means we can educate not only the clever and the average, but a large number of those who may be classed as stupid. But is it suitable for all boys, however stupid? I think not. And I should make Proposition 4 of Euclid the test; if a boy cannot by *any* means be brought to understand this Proposition, he had better devote his time to some other study than Geometry, however "New." For such boys, be they many or few, I should be disposed to recommend a little logic.

"Nothing is better than wisdom."

"A crust of bread is better than nothing."

"Therefore a crust of bread is better than wisdom."

Any boy can be got to see that, and to see it in some sort of a way where the fallacy lies; and his brains would be better exercised on this sort of thing than on a geometry which he was incapable of receiving.

I may sum up the results of this paper as follows:

1. That in deductive geometry, a definitely recognised standard of order is expedient.
2. That the Mathematical Association should consider whether any steps can be taken which will bring about the recognition of a standard order.
3. That the New Geometry makes it possible for a larger number of boys than formerly to benefit by the study, while it need not, if properly taught, diminish the mental training afforded.
4. That there are boys who do not profit by the study of any deductive geometry; such boys might learn some logic instead; but some training in practical geometry should be given to all.

Mr. Wagstaff concluded with the remark that it would be more interesting if there could be a stand-up fight between the supporters of the New Geometry and those violently opposed to it, and said that he had received a letter from Mr. Baker of Cheltenham, expressing the view that the experimental stage was being overdone.

Dr. Macaulay said that personally he was dissatisfied with the actual changes in the new books on Geometry. He did not know whether he could recollect the things to which he objected most, but one of them was this: far too much time is devoted to practical work. There is a whole section of practical work at the beginning, which is right: then begins a new kind of Geometry, namely, deductive geometry. But in the new books it was not true deductive geometry, for half of it was practical work. He thought such work ought to form but a small part of a course of deductive geometry: occasionally a practical question could be given, but they should be few and far between. It seemed to him that the books so far produced were without definitions and without axioms. Looking at one book recently, he read in one of the first propositions: "Which is impossible, otherwise two straight lines would enclose a space." That axiom had not been stated anywhere in the book. Such an axiom ought to have been carefully explained beforehand, since more than one meaning may be attached to it, although it is generally assumed to mean that two straight lines cannot intersect in two points. What we really want is an improved "Euclid"; we want Geometry made simpler, we do not want it made harder. He admitted it was almost impossible to produce such a Geometry. They wanted a book which was at least as logical as "Euclid." In the New Geometry, they had not yet got half the logic that was in "Euclid."

Mr. E. M. Langley said that Prof. Lodge had suggested to him that he had been attacked. He did not in the least look on Mr. Wagstaff's paper as an attack on him, for in all that he had said he had laid stress on the importance of logical training. The teacher must pay attention to strict logic. If the training is to be really practical, it must be as logical as possible, for a pupil who only learned an unconnected set of 'practical' devices would break down when he came to apply them. He thought it often did not matter what principle or axiom was assumed, if the deductions from it were rigorously made. Under the new system, we do succeed in getting the boys to reason about things whose nature they understand; under the old, we got them not to reason, but to reproduce—sometimes—the reasons given by Euclid from memory. Memory, however, was not to be so much decried as was sometimes done, and should be trained. A good memory would always be valuable. Let them fancy what it would be to deal with the construction of the icosahedron he had been speaking of, if we had forgotten the propositions it depended on and had to go back to first principles. On the whole, in teaching he adhered to the order of Euclid—he hoped an improved Euclid. He took the liberty of striking out and putting in as he thought fit. He thought the great advantage of the new system was that the teachers were no longer fakirs of the great Alexandrian prophet, but could introduce problems of their own for school treatment with the probability of their being just as likely to be set in the examinations as the old book-work.

Mr. W. J. Dobbs said it was generally assumed that the teaching of Theoretical Geometry associated a long chain of argument in which each proposition forms a link, and for that reason it was supposed necessary to adopt some recognised sequence. Our main premise is wrong. Many propositions can be proved directly from first principles. It would be quite simple if, instead of a recognised sequence, we could agree to look upon a few important geometrical facts as fundamental. Proof from first principles would of course sometimes be cumbersome, and a proof should be acceptable if it depends upon one of these standard propositions about which we may all agree. The methods of proof should depend mainly on the possibility of motion of one plane upon another by rotation or translation, or upon a supposed folding of the plane of the figure. Proof by folding ought to be looked upon as one of the fundamental methods, and should be introduced at quite an early stage. It appeals forcibly to young boys. Indeed, as soon as a child can cut out a pattern from the folded edge of a sheet of paper, he can appreciate the idea of symmetry. It is quite unnecessary to insist on some standard sequence.

Mr. A. W. Siddons spoke as Secretary of the Committee appointed by this Association. He addressed himself, first of all, to the question about standard order. If they were to have the best possible order, they must *not* have a standard order. If they had such a standard order, no one would attempt to improve on it. He quite appreciated the difficulty of having no definite order, but was there not possibly a natural order? Quite recently, four people—teachers of Geometry—sat down to draw up a natural order, and the order drawn up by all the four, independently, was practically the same. The real crux was in the first 32 propositions of Euclid: there the difficulty comes in. The title of Dr. Scott's paper was, "Is the New Geometry worth keeping?" They might add, "Are the teachers worth keeping?" for many of them were drifting off in a direction which some of the people who had to do with the change did not intend. He had taken some recent examination papers. The elementary questions are on Drawing, and the candidates would probably get through on the Drawing and the Propositions. The Riders were much too hard, and would probably not be attempted. So, you want to be good at Drawing and good at Propositions. If we want to encourage Geometry, we must not set too hard deductive questions: the questions must be within the pupils' limits. Teachers were, to some extent, being "choked off" deductions by examinations, for examinations were tending to set too hard deductions. Their Committee had inserted one important clause in their recommendations: "Pupils should do something besides mere book-work." They had asked for encouragement to teach something to develop deductive power. That encouragement was not what it should be. What was the object in teaching Geometry at all? They did not teach Geometry merely for the sake of Geometry. He regarded it as a training in logical reasoning. The great object of all teaching is to teach a boy how to learn. Many boys learn things at school and go to the Universities, and then find they do not know how to teach themselves. Dr. Macaulay had talked about too much practical work. Every new part of a subject is better introduced practically than theoretically, and also the facts are much more impressed on the boy's mind by their practical application, and so the end is gained by having a certain amount of practical work. Practical work is not an end in itself. He regretted that many examinations seem to be setting it up as such.

Mr. F. J. W. Whipple did not think enough stress had been laid upon the great difficulty there was in having a logical order. Anyone who attempts to teach boys, even boys who have had training in New Geometry, the first few propositions of Euclid, or propositions in the newer books, will see that it is in those few propositions that the difficulties occur. They were trying to make their boys of twelve prove propositions which were far more difficult than those propositions which boys learn later. They really wanted a break in the logical order, so that they could go on, come back afterwards, and then take up those difficult propositions. Take Euclid's eighth proposition. A complete proof was impossible: it would involve at least three previous proofs, and it would be very difficult to combine those three. The proposition is one that most boys are willing to take for granted, or to take as the result of a simple experiment. He held that the experiment would be the right proof in a case like that. The experiment made clear, too, a vast body of other knowledge, and they were able to show the connection between geometrical facts without troubling whether they had really got to first principles. That was the course pursued in every other branch of physical science. They did not attempt to find the first principles, but they tried to reach a knowledge of a great many facts, and, out of the great body of facts, to make a chain and correlate the separate facts. Therefore, it is not merely almost impossible to establish an order, but unwise. You show that one proposition can be deduced from another: you can usually reverse the process. If both propositions are equally well known, it matters not which is included in definition or axiom. Mr. Dobbs had entered a plea that the method of folding should be allowed. It was useful, but also very dangerous. It was difficult to make clear what assumptions were being made. Folding was an excellent experimental method: it was excellent in that it showed that the enunciation was plausible. But as a proof it was dangerous, because the boy, having once made his experiment, found it difficult to see that he had not completely proved all that he had been asked for. So, the possibility of a logical development of the subject was destroyed. But, by folding first, they could show that the proposition was plausible, and then, by using congruent triangles, give a sound proof. Reference had been made to the need for omitting some of the propositions quite at the beginning. There would be an enormous advantage, in that the necessity for introducing those propositions at the end would be emphasised. He had not seen a book on the New Geometry which really took up this question. Does a boy understand what he means by straight lines, by parallel lines? There may be a short note when parallels are first introduced, but there is no attempt at a systematic study of the difficulties of the question towards the end of the course. He had no doubt that many teachers did take some opportunity of doing that part of the work. The course in the text-books should include that part of the subject, and should give the teacher a better opportunity of developing it. He hoped that before the Association decided for the necessity of a logical order it would take due note of the difficulties to which he had drawn attention.

Mr. W. E. Paterson said there was one other point. Even more than a recognised order, recognised definitions were wanted. In the last few years there had appeared many different geometries. Many appeared to be quite logical in themselves. Of course many of the proofs must depend on the definitions. Parallel lines were variously defined and the different proofs have to be made to fit in with the definitions. There ought not to be much difficulty as to which were the best definitions; but of the definitions one met, there were few with which one more often than not had not to find fault. Many were redundant; he did not know whether he was using the correct term—they contained too much. For example, take the definition of a square: the old definition was that a square was a four-sided figure with all its sides equal and all its angles right angles. In the new books a square was often defined as a four-sided figure, having all its sides equal and one of its angles a right angle. Then they forget they have altered their definitions and make use, in a proof, of the fact that the square has *all* its angles right angles. With regard to Dr. Scott's paper, the title it had was given it simply to provoke discussion.

Mr. W. Fletcher said that no one in his senses would seriously propose that a boy who could not learn geometry should be taught logic instead. As an extreme case, proposition four had been mentioned. If a boy could not learn proposition four, he had better give up Geometry altogether, they were told. Now, there were some things which must be assumed or taken as the outcome of practical experience. The perplexities of the beginnings of subjects can be made endless. To keep a boy occupied at the beginning of Geometry until he has a clear conception of what straightness really is, or what parallelism really is, would be just as sensible as to introduce him to the fundamental difficulties of "What is time?" "What is space?" Most boys know whether a line is straight or not, or they *think* they know, and that is all any of us know of everything that we know: most boys know whether a line is parallel to another or not. The more one tried to explain proposition four to boys, the more perplexed the boys become. If you constructed the triangle step by step, the boy realised that the triangle is made. One had to be thankful for that, and he, for his part, had never yet come across the boy who could not understand that: but he would ask, Why should one not build on this foundation instead of puzzling the boy with abstruse questions as to the soundness of the foundation?

Professor Forsyth wished to thank Mr. Fletcher for his remarks. He confessed that he shared the surprise which Mr. Fletcher had expressed concerning one particular recommendation. It was a great surprise to find any one holding an opinion that, if a boy could not be made to understand the fourth proposition in the first book by a combination of practical illustrations and of reasoning, the boy should be turned on to logic. For such a boy with such a mental state, logic would be nothing less than a soul-destroying machine. In early training, pure logic must be kept in its proper place, which is not a prominent one at that stage: and it is best introduced through a medium in which it can be applied, and the speaker thought Geometry an exceedingly good medium for the purpose.

In regard to practical geometry, he was of opinion that in recent times some examination papers had undoubtedly contained too much of what might be called the practical element. It must be remembered though, that they who had urged the association of some practical element with the teaching of Geometry had had for a long time to fight in the wilderness. Only recently had they been allowed to associate the practical element with the other: and it would not be allowed to develop to an improper extent. The proper place of the practical element in Geometry was in providing the boy with the introduction to Geometry. It is undesirable to give him an elaborate mass of definitions and of axioms concerning things of which he had no knowledge whatever; but it is desirable to give him a comprehensible beginning, and then let him be turned on to some reasoning about the things he did understand from his practical work. He would give them an analogy, though he was aware of the danger of most analogies. It was like the teaching of modern languages. In older days, a boy was made to begin, generally, with the alphabet and with the aridities of grammar. But in order to grasp a language, it is better to know the words of the language first, even if they are put together in a wrong order, and then it is possible to pass to the grammar, the practice of which will teach what is to be done with the words. So, in Geometry, the best way is to begin with the practical part, so that a boy has something to go upon.

A good deal has been said about logic and logical order. But the great object of the recent changes was to get rid of one universal logical order; and if one order is to be constructed and mechanically imposed upon all, he doubted whether they could get a better order than Euclid's. The efforts of the reformers had been directed against the iron constraint of one single and uniform logical order; an iron constraint which prevented freedom, and which, he thought, owing to its very mechanical uniformity, often hindered a sound apprehension of the subject. And he would add that Euclid is only disestablished, not abolished.

In the course of the discussion an opinion had been expressed that it is desirable to have one uniform complete set of definitions, and that such a set of definitions could easily be constructed by common agreement. As the advocate of definitions spoke, he (Prof. Forsyth) thought of the many definitions of parallel lines, and of the controversies which had been waged thereon, and which in their fierceness and spirit had been not unworthy of those of theologians. He trusted that neither this Association nor any body of teachers would ever attempt to re-institute and re-establish one definite order of teaching in Geometry to be mechanically imposed on all. Teachers now have their freedom; let them value their freedom, even if it brought a little chaos. Some of them had recognised that probably at the beginning there would be times of difficulty, of chaos and of confusion; but they must all recognise that before they could make the amplest use of their freedom, they had had to pass through the valley of trouble.

One other thing, Mr. Wagstaff at the beginning of his paper had quoted the regulation of the University of Cambridge as to a proof that may be accepted by the examiners: the proof was to be one that satisfied the examiners as belonging to a systematic treatment of the subject. Now, what more could be asked for than that the boy shall be able to give a proof that seems to the examiners—qualified men and men of experience—to show a systematic treatment of the subject? Only through systematic treatment will that training of the mind, which was their object in teaching the subject, result. The full system is not yet in working order. Critics say there may be differences of opinion on the part of examiners: one examiner may hold different views from another. That is nothing new. Again, it has been said, examining has been made more difficult for the examiners. But the object of training is not the ease of the examiner; and if an examiner finds his work a little more difficult let him bear his burden cheerfully: he is exercising a useful function in a better system.

Then they had been told that if there be no universally recognised sequence, there will result, instead of teachers trying to give a systematic treatment of the subject, a tendency to train in dodges that make for fudging and other intellectually bad practices. He thought that an unworthy suggestion, and one that, he trusted, would not be accepted by any teacher of mathematics.

He did not think that justice was done in Mr. Wagstaff's statement to the Cambridge scheme: for that statement, in quoting only the kind of proof which would be expected, was seriously incomplete and required to be supplemented. The subject matter had been set out in two schedules—one containing the theoretical propositions to be proved, and the other showing the more important practical problems. Now, in drawing up those schedules very considerable care was exercised, and every proposition was subject to a very considerable challenge; propositions were deliberately put with their converses, so that whatever system had been adopted in a systematic treatment of the subject, a proof of the proposition was not limited to that contained in one particular system. It would only happen, perhaps, that the boy was proving first, in the system adopted for him, a part of the theorem that came later in a proof belonging to some other system.

In conclusion, he *did* consider there had been made possible an improvement in the teaching of Geometry in schools. It must be remembered that the examination, whatever worth it might be, was only an incident in the training, and the training was the matter that was of importance. If a teacher were looking forward to the examination only, he was giving attention to not the most worthy part of education. Teachers who, for examination purposes, neglected other parts of training, were neglecting the most important part of their teaching.

He apologised for the length at which he had treated the subject; but it was a subject that had occupied much of his attention, particularly in recent years, and his interest in it had not ceased.

Mr. Wagstaff, in closing the discussion, said he had been interested in it, and especially by the remarks of those who had attacked him with considerable vigour. He would not attempt to defend himself. He did not intend the suggestion of logic as an alternative to geometry to be taken so seriously. He would adapt his logic to the class before whom he proposed to place it. He did not think he differed from the gentlemen of the opposite side quite so much as they seemed to imagine. He thanked them all for the attention with which they had listened to his paper.

Mr. A. W. Siddons opened a discussion on the proposed new regulations *for the Previous Examination* at Cambridge. It might be expected, he said, by some present that his line of argument would be that the time which elapses between the scholarship examination and the date of going into residence at the University was an extremely valuable time for the mathematician and scientist; they could develop quietly without having any definite examination in view; but so long as they had to cram Greek for the "Little go" this time was almost wasted. He admitted that there was some truth in this argument, but he did not feel that the burden was really great for any boy who was capable of getting a scholarship, and he strongly urged that there should be no special relief for the mathematician. The boys who really needed relief were the average boys on the modern sides of our schools, boys who had had a good modern side education but who were not perhaps specially gifted with the power of acquiring classical languages, to whom the cramming of a mere smattering of Greek would be no training whatever and to whom such cramming would only give a profound contempt for the Greek language. After quoting from a recent statement by the Master of Trinity, Mr. Siddons urged all members of the Senate to go up and vote when the day came.

In the discussion on the question of compulsory Greek, **Mr. Heppel**, said the question was generally considered in the first instance as to what was to the advantage of all the growing youth of the country. There were some considerations that came before and besides that. A great many people entertained the idea that the word "University" had to do with the spread of knowledge over a great variety of subjects, and that one ought to be able to obtain instruction in any subject one wished. The word "Universitas" in its Latin legal signification meant simply "Corporation," and it was applied at the present day to those corporations which had to do with education and training. He did not think it a wise thing that every University should think it its duty to go over the whole field of knowledge. He did not want to see Oxford and Cambridge competing with new Universities like Birmingham, and he thought those newer Universities which were springing up, and he was glad to see them increasing, would do better work if they confined themselves to subjects likely to be of use to their members. Oxford and Cambridge had eminently distinguished themselves for Classical and Mathematical training. It was not as if the youth of the country could not get the other things. He had every hope that the universities now springing up would do noble work, and that they will do it more completely by devoting themselves to their special subjects. They would do the scientific work, and the work not connected with the ancient languages. He thought the boys who at school had some Greek taught them, might think it a bore; but he could not think that, when they grew up, they would ever regret it. The training in the language was something that lasted. At the last occasion when that question of Greek was raised at Cambridge, he himself was astonished at the number of mathematical men who decided for the retention of the old system. They felt that something had been introduced into their minds, something to enlarge their ideas which they could never have had without a study of Greek.

He had been interested to observe that the reader of the paper did not demand any special examination for mathematicians. With that he agreed. If Greek *were* to be compulsory, let it be compulsory all round, and if removed, let it be removed all round. He did not think it would be wise to remove it specially for mathematicians.

Mr. C. A. Rumsey said that it concerned the honour and credit of Cambridge University that the present state of things should be altered; that under the existing arrangements men were able to pass in the Greek required of them in the Little-Go by getting up a little grammar and one or two books with the help of cribs; even then it was only necessary to obtain 33 per cent. of the total marks, and on this performance men were hall-marked by the University as having a knowledge of Greek. It was quite usual for the requisite knowledge to be acquired in six weeks, and he believed the record was thirty hours. The Little-Go was, as a matter of fact, only a part of the general examination for the B.A. degree, from the remainder of which it had been decided to excuse those who obtained honours in a Tripos. Now this examination involved a definite knowledge of Latin and Greek, including the power to do unseen passages, and this was the least which ought to be demanded if any kind of certificate was to be given. He therefore advocated one classical language instead of two, with a standard up to translations of unseens. Unseen passages were required at London Matriculation, which really guaranteed a knowledge on the part of candidates of the languages in which they passed. The only satisfactory solution of the Cambridge question was to allow the choice of one or two classical languages with leave to substitute a modern language for one of them if so desired.

Mr. W. D. Eggar said that if as a visitor he might be allowed to speak, he would like to ask a few questions. Some people said the study of Greek would become extinct if these recommendations were adopted. He wished to know what the opinion was about that: were there strong grounds for saying Greek would *not* become extinct? Then, perhaps, Professor Forsyth would tell him what objection there was to two degrees—that of B.A. and, say, some such degree as B.Sc., though that seemed an unpleasant alternative. Still, it did seem desirable, from some points of view, to give some encouragement to Greek. A great many schoolmasters held that Greek should be begun by boys at 15, and not in the preparatory schools; and it was said that if this course was adopted, and if Cambridge abolished Greek, hardly any boys would take it up at the public schools.

Mr. C. S. Jackson, as a non-resident member of the Senate, urged the settlement of the matter by the people on the spot. He regarded it as a monstrous thing that a man, who had perhaps ceased for twenty years or more to have any concern with educational matters, should take upon himself not only to vote for the retention of Greek, but to agitate and urge others to do so in total disregard of the well-considered opinion of the eminent men who were, so to speak, the managing directors of the University. He protested against it, and thought the least such non-resident members could do was to stay away.

Prof. Hudson was by no means certain that if compulsory Greek were abolished, Greek would be destroyed. The proposal was that *either Latin or Greek* should be demanded. No dire result would follow—relief would only be given to a small number of students. The main argument in favour of the change was that it would tend to destroy the undue specialism in language which was the bane of the education in this country. This was only a very little move in the opposite direction.

Prof. Forsyth said an appeal had been made to him to answer one or two questions, as he happened to be a member of the Cambridge Syndicate dealing with the matter. He would reserve for other occasions remarks of a general kind, and would address himself to one or two points. Mr. Jackson had stated as his opinion that non-resident members should not come and vote. But the appeal had been made on both sides in the controversy, and they had to work under the constitution of the University: no decision could be made by the University as a body without the consent of the Senate. The sole power rested with the Senate. Most of the questions that came up were, of course, settled readily, and outsiders did not come and vote. An appeal had been made to the Senate on this question, and, as members on the other side would go up and vote, he hoped that those in favour of the proposals would go also.

With regard to the question of the "previous Examination," he did not know whether those present would be surprised to hear that Greek was made compulsory as lately as in 1822. They were not dealing with something that had come down to them from the Middle Ages. The regulations were comparatively recent, and some of them were of opinion that, for grave reasons, the time had come for modifying them.

He would give them two other pieces of history. Down to 1850, every man who wished to be qualified for Honours in Classics was required to have obtained Honours in Mathematics; that is, to have passed the Mathematical Tripos. Classics, then, was the persecuted subject. Macaulay, for example, could not take the Tripos: as he said in a letter to his mother that it was impossible for a man to be interested in the formula for $\tan(a+b)$, which he quotes pathetically and incorrectly; he failed to pass the Mathematical Tripos, and took only a poll degree.

Again, down to 1871, only those were eligible for the Chancellor's Classical Medals who had not merely passed the Mathematical Tripos, but had attained the standard of a second class.

The proper changes were adopted, and the exaction of Mathematics was no longer made from Classical students. But he was not aware that, when the changes were suggested, mathematicians arose in the might of their numbers and declared: "This change must not be made; if it is, the study of mathematics in twenty years' time will be extinguished in England": or that they made any similar cheerful prophecies. And so he did not think people were justified in saying that, because this miserable minimum of Greek which is imposed in the "Little-go" would be allowed to be optional, therefore the subject would die out. The subject of Greek had a strong hold in the country. Now, it was four or five times as flourishing as it was fifty years ago. The teachers are more in number; the scholarships and fellowships in the ancient Universities are still open. It could not be damaged by the non-requirement of this miserable modicum, which any fairly intelligent boy could get up in a term.

It was a strange attitude of mind to think that Greek would become extinct, because a number of boys would not take the Greek that is in the "Little-go." He acknowledged that in a large number of the schools in the country Greek is already extinct; but that was owing to the action of the Charity Commissioners, who had drawn up new schemes for the schools, omitting Greek from those schemes. He thought, but was not quite certain, that in cases where Greek was omitted from the new scheme for a school, it had not been taught there at all before the new scheme was imposed. Even so, is this fact to be altered in the least and is the study of Greek to be secured by the conditions of the Previous Examination? It is impossible to believe that the conditions about Greek would affect these schools at all. If the Universities retain these regulations, either boys are to be cut off from the Universities or they get up the subject—from the point of view of educational training? No;—from the object of gaining a knowledge of

Greek life, thought, and literature? No; but from the point of view of satisfying examiners with a miserable modicum of Greek that, in many cases, gives no education and implies no knowledge worthy of the name. In the proposals of the Syndicate, Greek is obtained and still could be taken by those who had something more than a scrap of knowledge; but it would not be imposed upon all, and, having regard to the schools up and down the country, he contended that the changes were by way of furthering the improvement of education in the country.

Sometimes he was challenged as to whether he knew of Greek having done any harm. He could not say he knew of cases where it had done much harm; but he did know of cases where it had done some harm, and many in which it had done no good. He did know of cases where boys of ability had got up this Greek in a single term, being diverted from their special studies, and he had heard, time after time, of young men who regretted—even resented—having been compelled to get up a bit of this language for such an examination as the Little-go was. It had not provided training or knowledge. It was a test that was simply a survival; and it had come to be an exceedingly low test.

He had heard of boys who had spent many years at Greek and never been able to get hold of it, or anything of Greek; they did not learn the humanities, they never grasped the spirit of Greek tragedy; they did not get to know anything of the spirit of Greek literature: those boys were very frequent. Not merely did they not get their minds trained, but they came up to the University without any appreciation of Greek literature, sometimes animated by a positive resentment against it, sometimes even with a repulsion to all things intellectual. He trusted all who wished to do something to make things better would go to Cambridge and support the proposals of the Syndicate.

Mr. Eggar had asked: "Why not two degrees—B.A. for people who take Greek, B.Sc. for people who don't?" Well, there was something to be said for it, but the matter was not so simple as it looked at first sight; and further consideration would reveal difficulties and objections about the suggestion when it is made in reference to present Cambridge conditions. They might have a man who was in the First Class of a Tripos who had not taken Greek in the Little-go, and another in the Third Class who had taken the Little-go Greek. Was the first to get the degree of B.Sc. and the other the degree of B.A.? Such a result would not be just. Then it had been suggested that they might institute the degree of B.Sc. for those who know no Greek, but do the ordinary course, and that of B. of Letters for those who know no Science. But the same difficulty came up again when they came to look into the details. Another suggestion was that it might be possible to institute new degrees—B.Sc. mainly scientific, B. Letters for a course mainly literary, and B.A. for a course that involved both Science and Letters.

Of course it would be possible to devise schemes for new degrees that were not dominated solely by the presence or absence of Little-go Greek. But he wished to point out that, if new degrees were instituted which did not involve a knowledge of this minute amount of Greek, then for all who took such degrees Greek would have become optional merely by another way; and those who feared disaster to Greek would presumably be opposed to this project also. Moreover, he could not help noticing that the suggestion of a new degree never was made except when the question of Greek is raised, and he submitted that it was not for those who did not ask for new degrees, but did ask for something else, to construct schemes for those new degrees.

What they of the Syndicate were asking for was a relaxation in favour of education in general, and not for a particular class of people. The language to be taken, if Greek be dropped, must be a modern language, which must be taken up to translation, without the help of a dictionary. The Syndicate had considered the main reasons which ought to underlie the general scheme of culture and education, and they were asking for the changes on behalf of the great bulk of young men who went up to the University. The speaker concluded by asking all those who agreed with the proposals of the Syndicate to go to Cambridge and vote for them when the day for voting should have come.

THE DEVELOPMENT OF GEOMETRICAL METHODS.

XI.

Here, again, in this infinitesimal branch of Geometry we find the two tendencies which I pointed out in connection with the geometry of finite quantities. Some, among whom we must place J. Bertrand and O. Bonnet, endeavour to construct an autonomous method directly based on the use of infinitely small quantities. Bertrand's great *Traité de Calcul différentiel* contains several chapters on the theory of curves and surfaces which, in a measure, illustrate this conception. Others follow the usual analytical paths, and content themselves with clearly recognising and throwing into relief the elements which should figure in the foreground. This is what Lamé did when he introduced his theory of *differential parameters.* Beltrami followed this lead when he extended with great ingenuity the use of these differential invariants to the case of two independent variables, *i.e.*, to the study of surfaces.

Just now there seems to be a return to the mixed method, the origin of which is to be found in the works of Ribaucour under the name of *perimorphie.* The rectangular axes of Analytical Geometry are retained, but they are moveable, and connected with the system under discussion in whatever appears to be the most convenient manner. In this way most of the objections that can be levelled at the method of co-ordinates disappear. Thus are united the advantages of what is sometimes called *Intrinsic* Geometry to those which result from the use of regular analysis. But, this analysis is by no means abandoned. The complicated calculations which it almost always involves in its application to the discussion of surfaces and of rectilinear co-ordinates disappear in most cases if we utilise the conceptions of the invariants and co-variants of the quadratic differential functions which we owe to the researches of Lipschitz and Christoffel, inspired by the work of Riemann in non-Euclidean Geometry.

XII.

The results of such investigations are almost apparent at once. The notion that Gauss already possessed of geodesic curvature, but which he had not published, was given by Bonnet and Liouville; the theory of surfaces of which the radii of curvature are functions the one of the other, inaugurated in Germany by two propositions, which are worthy of a place in the pages of Gauss's memoir, has been enriched by Ribaucour, Halphen, Sophus Lie, and by others with many theories. Some relate to these surfaces regarded from the general point of view. Others are applied to the particular cases in which the relation between the radii of curvature takes a peculiarly simple form—to minimal surfaces, for instance, and also to surfaces of constant positive or negative curvature.

Minimal surfaces have been the object of investigations which form the most attractive chapter in infinitesimal geometry. The integration of their partial differential equations constitutes one of the finest discoveries of Monge; but, in consequence of the imperfect condition of the theory of imaginaries, the great geometer was unable to deduce from his formulæ any method of the generation of these surfaces, nor even of any particular surface. I shall not here return to the detailed historical sketch which I have already given in my *Leçons sur la Théorie des Surfaces*; but it is convenient to recall Bonnet's fundamental investigations, which have given us, in particular, the idea of *surfaces associated with a given surface*; Weierstrass's formulæ, which established the closest relationship between minimal surfaces and the functions of a complex variable; Lie's researches, in which he showed that Monge's own formulæ may now serve as a foundation for a fruitful survey of minimal surfaces. By seeking to determine the minimal surfaces of very small classes or degrees we are led to the conception of the double minimal surfaces which we obtain in the *Analysis Situs*.

In this theory three problems of unequal importance have been considered. The first, relating to the determination of minimal surfaces inscribed in a given contour in a developable which is also given, has been solved by well-known formulæ, which have led to a large number of theorems. For example, every line traced on such a surface is an axis of symmetry.

The second problem, due to Sophus Lie, relates to the determination of all the algebraical minimal surfaces inscribed in an algebraical developable, when the curve of contact is not given. It has also been completely solved.

The third and most difficult is that which physicists solve by experiment, plunging a closed contour into a solution of glycerine. They have then to determine the minimal surface passing through a given contour.

The solution of this problem is clearly beyond the resources of geometry. Thanks to the resources of the most advanced analysis it has been solved for particular contours in Riemann's celebrated Memoir, and in the profound researches which followed or accompanied it. As for the most general contour, its discussion has been brilliantly commenced, and will be continued by our successors.

Next to minimal surfaces, surfaces of constant curvature were certain to attract the attention of geometers. An ingenious remark of Bonnet's connects both the surfaces, in which one or the other of the two curvatures, the mean or total curvature, is constant. Bour had asserted that the partial differential equation of surfaces of constant curvature could be completely integrated, but this does not appear to be the case. It even appears more than doubtful, if we refer to a discussion in which Sophus Lie tried in vain to apply a general method of integration of partial differential equations to the particular equation of surfaces of constant curvature. But, if it is impossible to determine in finite terms all these surfaces, at least some can be obtained characterised by special properties, such as that of having plane or spherical lines of curvature; and, by the use of a method which has succeeded in many other problems, it has been shown that from every surface of constant curvature can be derived an infinite number of other surfaces of the same nature, by clearly defined operations which only require the calculation of areas.

The theory of the deformation of surfaces in Gauss's sense has also been enriched. To Minding and Bour we owe the detailed study of the special deformation of ruled surfaces which leaves the generators rectilinear. If we could not, as I have just said, determine surfaces applicable to the sphere, at any rate we have attacked with some success other surfaces of the second degree, and, in particular, the paraboloid of revolution. The systematic study of the deformation of general surfaces of the second degree is already commenced. It is among the researches which in the near future will give very important results.

The theory of infinitely small deformation is now one of the most complete chapters in Geometry. It is the preliminary and slightly extended application of a general method which seems to have a great future before it.

Given a system of differential equations, or of partial differential equations suitable for the determination of a certain number of unknowns, it is convenient to associate with it a system of equations which I have called an *auxiliary system*, and which determines the systems of solutions infinitely near to any given system whatever of solutions. The auxiliary system being necessarily linear, its use in all researches throws valuable light on the properties of the system proposed, and on the possibility of obtaining its integration.

The theory of lines of curvature and asymptotic lines has been notably extended. Not only have these two series of lines been determined for particular surfaces, such as Lamé's tetrahedral surfaces, but also by developing Moutard's results relative to a particular class of linear partial differential equations of the second order, we have been able to generalise all that had been obtained for surfaces with plane or spherical lines of curvature, by completely determining all the classes of surfaces for which the problem of *spherical representation* can be solved. In the same way, the correlative problem relating to asymptotic lines has been solved by recognising all the surfaces of which the infinitely small deformation can be determined in finite terms. Here there is a wide field of research in which exploration has hardly begun.

The infinitesimal study of rectilinear congruences, already begun long since by Dupin, Bertrand, Hamilton, and Kummer has intervened in all these researches. Ribaucour, who took a preponderating share in these investigations, studied particular classes of rectilinear congruences, and, in particular, those which are called *isotropic*, which intervene in the happiest manner in the study of minimal surfaces.

The triply orthogonal systems employed by Lamé in Mathematical Physics have become the object of systematic research. Cayley was the first to form the partial differential equation of the third order on which the general solution of this problem had been made to depend. The system of confocal surfaces of the second degree has been generalised, and has given birth to the theory of general *cyclides*, in which we may employ the resources of Metrical Geometry, Projective Geometry, and of Infinitesimal Geometry. Many other orthogonal systems have also been discovered. Among them it is worth while to refer to the *cyclical* system of Ribaucour, in which one of the three families has circles as its orthogonal trajectories, and the more general systems for which these orthogonal trajectories are simply plane curves. The systematic use of imaginaries, which we must be extremely careful not to exclude from Geometry, has enabled us to connect all these determinations with the study of the finite deformation of a particular surface.

Among the methods which have enabled us to establish all these results, we should note the systematic use of linear partial differential equations of the second order, and the systems formed of such equations. The most recent researches show that their use will revolutionise most of the theories.

Infinitesimal Geometry was certain to affect the study of the two fundamental problems of the calculus of variations.

The problem of the shortest path on a surface was treated in the masterly papers of Jacobi and Ossian Bonnet. The study of geodesic lines has been followed up, and we have learned how to determine them for new surfaces. The theory of aggregates has intervened and has enabled us to follow these lines in their course on a given surface. The solution of a problem relating to the representation of two surfaces, one on the other, has greatly increased the interest of the discoveries of Jacobi and Liouville relative to a particular class of surfaces of which we can determine the geodesic lines. The results in this particular case led to the examination of a new question—the discovery of all the problems in the calculus of variations, the solution of which is given by curves satisfying a given differential equation.

Finally, the methods of Jacobi have been extended to space of three dimensions, and have been applied to the solution of a question which presented the greatest difficulties—the study of the minimum properties of a minimal surface passing through a given contour. *(To be continued.)*

MATHEMATICAL NOTE.

164. [K. 21. 1.] *The Constructions in Halsted's Rational Geometry.*

There are some very curious resemblances in the paper by F. S. Macaulay, in the October *Gazette*, to a book which appeared last summer, and these resemblances are scarcely lessened by the perpendiculars used, after the first problem, (1), identical with § 164, being supposed drawn in the *Gazette* without, in the book with, use of parallels, especially as in the problems, *e.g.* (6), identical with § 163, no mention of this is made.

Not only the constructions, but more noteworthy still, the instruments are identical, for the *Gazette* uses the ruler and sect-carrier of the book, though not calling it by Halsted's name, since in fact Dr. Macaulay's article was in type for the July number of the *Mathematical Gazette* and was crowded out till October number, while the book reached the author at the earliest in late September.

I have since learned that the fundamental construction, (1) ≡ § 164, was given by R. F. Muirhead in a "Substitute for Euclid's Third Postulate" (Proc. *Edinburgh Math. Soc.* 1880).

In (2) the *Gazette* solves by using an angle-bisector, a ruler and a sect-carrier, the problem: To draw a perpendicular to a given straight. This it then applies to (3), or by means of a set-square, an angle-bisector, a ruler and a sect-carrier it solves the problem: To draw a perpendicular to a given straight from a given point on it.

Had the book so wished, it could have solved both problems together, not only without parallels, but without set-square and without sect-carrier, with only an acute-angle-bisector and a ruler.

GAMBIER, OHIO, U.S.A. GEORGE BRUCE HALSTED.

REVIEWS.

The Dynamics of Particles and of Rigid, Elastic, and Fluid Bodies; being Lectures on Mathematical Physics. By ARTHUR GORDON WEBSTER, A.B., Ph.D. Professor of Physics, Director of the Physical Laboratory, Clark University, Worcester, Massachusetts. Teubner, 1904. Pp. xi, 588.

The aim of this book, as stated by the author, is to give in a compact form a treatment of so much of the Science of Dynamics as should be familiar to every serious student of Physics. The range covered is a fairly wide one, including, in addition to the subjects indicated by the title, investigations of mathematical results which may be supposed to be beyond the reader's knowledge, such as theorems in curvilinear coordinates, properties of Spherical Harmonics, and the reduction of quadratic forms to sums of squares such as is needed for the establishment of the normal coordinates of a vibrating system. The result is a book which has considerable merits, and which gathers together a great deal of useful theory. Considerable portions of it are, however, not new, but have been reprinted from the author's treatise on Electricity and Magnetism.

From the points of view of most students of Physics, the book is open to the criticism of being overloaded with mathematical analysis. Conciseness is not always an advantage, but it must be right to state things that are really simple as simply as possible. Professor Webster is too apt to envelop them in a maze of symbols. Take the case of Lagrange's equations. The investigation of them is unduly long, and the case in which the kinetic energy is a homogeneous function of the velocities is dealt with without a clear statement of the condition under which this is true; and, finally, the removal of this condition, though the point which it involves is a very simple one, occupies a page and a half of rather repulsive appearance. To take another case. The component velocities of a point in terms of coordinates referred to moving axes are found by means of the whole array of the direction cosines of these axes and the relations between them; a lengthy procedure, though it may be regarded as a neat piece of mathematics, if that is what interests us. In the course of the work we find that the point is supposed to be fixed to the moving axes, which is not specifically mentioned except in a subsequent article. Eventually the usual expressions in terms of angular velocities are obtained. But the fact that these are almost obvious, and that nobody whose sole object was to obtain them would think of adopting such an intricate process, is not alluded to. In some similar cases the existence of a possible short cut is mentioned, but more often this is left for the reader to discover, if he can, as a pleasing surprise, when he realizes with sufficient clearness the meaning of the result obtained. The investigation of the motion of the centre of mass of a system is a curious example of the author's worst manner. For the space of two pages, including some mathematical analysis which as it stands is not strictly correct, we are led to suppose that the result is connected with the existence of a force function; and on the third page this restriction is removed. Incidentally we are told that the centre of mass of the solar system is evidently a case of a point moving with constant velocity in a straight line. The references to the solar system are not very happy, for elsewhere we find the remark that Kepler's laws give a complete kinematical statement of the law of gravitation for the planets, and we have to read on for ten pages before we meet with any correction of this view.

A proof which is given of Stokes's theorem by means of variations, said to be derived from Helmholtz's lectures, is a neat piece of mathematics; and if Professor Webster here errs in proving the result in an indirect and artificial fashion, he does so in good company. Why is it that nobody seems to think that a straightforward integration of the double integral with regard to x and y coordinates, such as the following, is good enough?

Let u, v, w be the components of a vector, u_1, u_2, u_3 the differential coefficients of u with regard to x, y and z, and l, m, n the direction cosines of the normal to a surface of which dS is an element of area; then we start with

$$\iint\{l(w_2-v_3)+m(u_3-w_1)+n(v_1-u_2)\}\,dS.$$

Take the terms involving u, and x and y for the two independent variables for this part of the integral, which accordingly is

$$\iint\left(\frac{m}{n}u_3-u_2\right)dx\,dy.$$

Now $-\dfrac{m}{n}$ is $\dfrac{dz}{dy}$ on the surface, thus this integral is $-\iint\dfrac{du}{dy}\,dx\,dy$, or $-\int(u''-u')\,dx$, that is to say $\int u\,dx$ taken round the boundary in the direction which makes with the normal a screw of the same sign as that which a rotation from x to y makes with the axis of z. Thus the theorem is proved for any portion of surface for which projection on the three coordinate planes does not involve any complication. But applying the theorem to an element of surface, it is clear that we have proved that if the vector u, v, w is a definite one independently of the choice of axes, w_3-v_3, u_3-w_1, v_1-u_2 are components of a vector which has the same property. Thus considering the integrals as a line integral of one vector and surface integral of another without reference to axes, also the fact that both can be added for contiguous portions of a surface, we see that, with the necessary restrictions as to continuity, the theorem applies to any case of a portion of surface which is bounded by a closed curve and has two distinct sides.

Professor Webster has done useful service by setting the example of a free use of the term lamellar, applied to a vector distribution which has a potential. Except for its use to specify a laminated structure of magnetization, the word has not been generally adopted. But no better adjective to describe the thing has ever been proposed, and it illuminates the subject in the same sort of way that the introduction into physics of the word vector did. What is wanted is an adjective, and a neutral one, which suggests the geometrical arrangement and nothing more. Professor Webster also employs the old-established term solenoidal; but lamellar and tubular make a better pair, two Latin words instead of one Latin and one Greek. Besides, the word tube is in general use in the same connection, and solenoidal has been somewhat spoilt to the ear by the use of the term solenoid in practical electricity in a totally different sense. Professor Webster also does well to use the term vector product. Unfortunately, Hamilton's scalar product of two vectors has an inconvenient sign, so the term geometric product is adopted instead with the opposite sign. This serves the purpose though not particularly suggestive. Another good word adopted is slide for *glissement* instead of the confusing word shear;

but there does not seem to be sufficient reason for the innovation of giving it half its usual value, considering how much confusion this may make. The same remark applies with less force to curl. The resuscitation of Lamé's first and second differential parameters is also of doubtful advantage; they are not suggestive names. But any way Professor Webster has a better notion of providing enough suitable names for the things he has to talk about than most writers on the subject. He might, especially in writing for physicists, make more use of the conceptions thus expressed to gain conciseness and vividness of statement, and to avoid cumbrous and unnecessary mathematical work; but it is clear that he does not care for conciseness, and that it is not his plan to take any course which could have the appearance of being a short cut, even though it may be one which has a good right to be regarded as the high road.

His account of Helmholtz's theory of vortex motion is a case in which he puts the reader on the right track at the outset: the point to which he calls attention here being the idea of analysing a vector distribution into two components, one lamellar and the other tubular. But he is surely wrong in attributing this analysis to Helmholtz. It was given in exactly the same form by Stokes nine years before Helmholtz's theory of vortex motion was published; see Stokes's papers, vol. ii, p. 254. Here the expression of the tubular component as a curl, that is to say the discovery of the vector potential, is an essential point; but apart from this, the fact that the analysis is possible may be very easily seen. Suppose any vector distribution to be represented in magnitude and direction by lines, forming a fibrous structure in which lines begin and end as is required by the variations of magnitude. Add to this what is necessary to make the system tubular by splitting up each line, at the point at which it ends, into radii uniformly distributed in all directions, and similarly drawing a system of radii up to the point at which each line begins. The system thus added, taken by itself, is lamellar; accordingly the original system is analysed into the tubular system which has been constructed and a lamellar system consisting of the added part reversed. The case of the analysis of 4π times magnetization into magnetic induction and magnetic force reversed is of course an example of this. But there is an advantage in the matter of clearness in getting the geometrical conceptions separated from their physical applications.

There is much in Professor Webster's book which is commendable for thoroughness. For example, in dealing with the flexure and torsion of prisms, we are plunged at once into St. Venant's theory, and are not allowed a chance of entangling the elastic line with any speculations about cross sections in flexure remaining plane.

The treatment of rotating bodies, including tops, and the author's own contributions to the experimental verification of such motions, deserve special attention. W. H. MACAULAY.

Eléments de la Théorie des Groupes Abstraits. Par J. A. DE SÉGUIER. (Paris, Gauthier-Villars), 1904.

This is the short but full first part of a contemplated work of which the master-title is "Théorie des Groupes Finis." The whole will clearly aim at completeness, so far as present knowledge goes. The present part is difficult reading because of extreme compression, and also because the order of arrangement has been decided by considerations of abstract thought rather than by naturalness of discovery. A principle usually adopted by French authors is that a work worth making public at all is worth writing in sufficiently expanded form, and printing with sufficient prodigality of space, to make it attractive and convincing to those to whom its ideas are new. Lucidity is put on a par with exactness, and to be lucid without being wearisome is to effect a triumph of art. On our side of the channel views as to artistic expression have too often been different. Not to use a word too many, not to waste two lines over an analytic expression which inventiveness in notation can confine to one, have been aims with which we are too familiar. We thus deter scores from gathering anything from what we offer, and comfort ourselves with the belief that the very few who do spend a considerable fraction of a life-time in ascertaining what we mean must really thereby master the subject dealt with, because practically forced to investigate the full content of every sentence for themselves. M. de Séguier seems to have discarded as vanity the pride of his compatriots, to have adopted our false canon of art, and even to have succeeded in outdoing us.

The word "Finis" in the master-title appears redundant. It hardly prepares us to find the preliminary chapter in the introductory volume devoted to infinite assemblages: to an arithmetic in which the two cardinal numbers having prominence are ∞ and ∞^{∞}. The finite is to be studied as a special variety of what is generally transfinite. The once intelligible word "Eléments" also does service in a sense even less elementary than has become usual. We have grown accustomed to a lax use of the term which we associate with diffidence on the part of authors. A comprehensive treatise is often quite unnecessarily, and even misleadingly, described on its title page as one on the elements of a subject, because it starts from the beginning, and the author is aware (as all must be who know what they are talking about) that he has not said the last word possible. But here the word marks no condescension to the beginner at all. Easily grasped ideas are crude, and savour of the concrete. The view has been adopted that it is unscientific to disclose how these have been analysed, and thought has passed back from them to more abstract ones which they involve. On the contrary, fix upon the utmost thinkable which has been passed back to, do your best to reduce it in full generality to the stateable, and call the result an elementary idea. Reason from it as you can, avoiding all acknowledgement of concrete fetters as means of holding it, and so develop what you may call the elements of a theory. Presently descend to things concrete, to the simpler ideas which will help the ordinary human mind in its efforts to grasp the "elementary" conceptions which have preceded. The descent is to be made in the next volume under the heading "Compléments."

The book is to be admired as consistent and masterly. The above remarks are not intended as adverse criticism—many will regard them as amounting to high praise—but as allegations in support of the present reviewer's opinion that had the author been less careful to avoid diffuseness, and less determined in his adherence to a lofty ideal of rational arrangement, he might have appealed to a wider circle of readers, and have supplied a want of which many are conscious, while still serving higher purposes. All who have thought about groups and aggregates, and want to possess in compact compendium what they know and should know, will welcome the present volume and desire the early appearance of the "Compléments." We share their feelings of welcome and expectation; but at the same time feel more strongly than ever that there is urgent need of a popularizing introduction to abstract group theory, a book which will aim in the first place at awakening, and only secondarily at satisfying, enthusiasm for the subject. E. B. ELLIOTT.

Die Lehre von der Zentralprojektion im Vierdimensionalen Raume. By Dr. H. DE VRIES. Pp. 78. (Leipzig, Goschen.)

The straight lines which join any point O to all points of a plane trace out a space of three dimensions, and, conversely, the properties of the space may be studied by means of its projection from O on the plane. Similarly, we may project a space of four dimensions from O on to a three-dimensional space and obtain its properties, though the process is naturally more complicated. The author works out this idea in detail, starting with Schoute's "Mehrdimensionale Geometrie" and Fiedler's "Darstellende Geometrie" as his basis. The first part of the book is occupied with a general discussion of the projection. the vanishing points of lines, etc.; then comes the theory of perpendicularity and a series of fourteen fundamental constructions, *e.g.* the line of intersection of a plane and a space, the distance of two points, and so on. H. H.

Mathematische Einfuhrung in die Electronentheorie. By A. H. BUCHERER, University of Bonn. (Teubner.)

The idea that the phenomena of electrolysis are produced by the transference of small portions of matter carrying an electric charge, is familiar to every student of physics. The discovery that in part of the phenomena of the vacuum discharge we are concerned with the motion of charged portions of matter which have a smaller mass than the ordinary chemical atom, has led to a profound change in our ideas of the constitution of matter. The attention of mathematicians has thus been strongly directed to the study of the motion of small charged systems in accordance with the electromagnetic equations of Maxwell and Hertz, for the question has an important bearing on spectrum analysis and radio-activity.

The problem is one of great difficulty and has not yet been solved in general. At present it must be regarded as in a tentative stage, and thus any book on the subject with any pretensions to accuracy, must involve a constant appeal to experiment.

Dr. Bucherer has been very happy in his treatment of the subject. By using vector notation he has reduced the mathematical symbolism considerably and has thus been able to present the results of mathe-

matical investigation and experimental research in an exceedingly compact manner.

The student will look in vain for a definition of an electron. The simplest idea is that of Larmor, in which the electron is regarded as a charged point. Difficulties, however, arise unless we ascribe to the electron a definite size, and this introduction again raises difficulties as to how the charge is arranged, and what constraints are to be applied. Such difficulties may well be avoided in a first study, and the book can be confidently recommended, not only to students, but also to experimental investigators who desire a compact account of the results of mathematical analysis. G. W. WALKER.

ERRATUM.

P. 111 for "in the Mathematical Tripos this year" read "p. 467 *Nouvelles Annales*, 1903."

COLUMN FOR "QUERIES," "SALE AND EXCHANGE," "WANTED," ETC.

(1) **For Sale.**

The Analyst. A Monthly Journal of Pure and Applied Mathematics. Jan. 1874 to Nov. 1882. Vols. I.-IX. Edited and published by E. HENDRICKS, M.A., Des Moines, U.S.A.

[With Vols. V.-IX. are bound the numbers of Vol. I. of *The Mathematical Visitor.* 1879-1881. Edited by ARTEMAS MARTIN, M.A. (Erie, Pa.)]

The Mathematical Monthly. Vols. I.-III. 1859-1861 (interrupted by the Civil War, and not resumed). Edited by J. D. RUNKLE, A.M.

Proceedings of the London Mathematical Society. First series, complete Vols. 1-35. Bound in 27 vols. Half calf. £25.

Cayley's Mathematical Works. Complete, equal to new, £10. Apply, Professor of Mathematics, University College, Bangor.

The Mathematical Gazette. Nos. 7-18 inclusive, £1. No. 8 is out of print and extremely scarce.

(2) **Wanted.**

The Messenger of Mathematics. Vols. 24, 25.

Tortolini's Annali. Vol. I. (1850), or any of the first eight parts of the volume.

Carr's Synopsis of Results in Elementary Mathematics. Will give in exchange: Whewell's *History* (3 vols.) and *Philosophy of the Inductive Sciences* (2 vols.), and Boole's *Differential Equations* (1859).

CORRESPONDENCE.

ELEMENTARY PURE GEOMETRY WITH MENSURATION.

E. BUDDEN.

To the Editor of the "Mathematical Gazette."

DEAR SIR,—The non-rigorous treatment of the perpendicular or right bisector and other loci, to which Prof. Lodge refers in his review of my book in the last number of the *Gazette*, occurs only in the experimental introduction. This is meant to be taken orally with the beginner, and is designed to familiarise him with the use of the instruments and with the figures most commonly required in formal and constructive geometry; it is not pretended that the results given in this introduction are formally proved.

In the formal geometry, which follows the introduction, the right bisector and other necessary loci are given quite rigorously; in each case it is shown (i) that every point of the locus is on a certain line or curve, (ii) that each point of this curve is on the locus.—Yours faithfully, E. BUDDEN.

OBITUARY NOTICE.

M. PAUL TANNERY.

We regret to learn of the death of M. Paul Tannery on November 27th ult., at the age of 61. M. Tannery, after studying in the Polytechnique at Paris, devoted himself professionally to the tobacco industry, and became director of the Pantin tobacco works. He was a man of varied interests, his writings dealing with mathematics, philosophy, the history of sciences, philology and Hellenic studies. In addition to papers on these subjects, he wrote treatises on Greek geometry and the history of astronomy, and collaborated in editing the works of Fermat and Descartes. He once gave a course of lectures on the history of Arithmetic at the Sorbonne, and also lectured on Greek and Latin temporarily in the Collège de France. Recently he has taken an active part in drawing public attention towards encouraging the study of the history of the exact sciences.

BOOKS, ETC., RECEIVED.

The Schoolmasters' Year Book and Directory. Pp. 558. 1905. (Swan, Sonnenschein.)

Lectures on the Calculus of Variations. By OSCAR BOLZA. Pp. 272. $4. 1904. (The University, Chicago Press.)

Introduction à la Géométrie générale By G. LECHALAS Pp. x, 58. (Gauthier-Villars.)

An Introduction to the Calculus based on Graphical Methods. By G. A. GIBSON. Pp. vii, 225. 3s. 6d. 1904. (Macmillan.)

Elementary Algebra. Part II. By W. M. BAKER and A. A. BOURNE. Teachers' Edition. 3s. Pp. 277 468. 1905. (Bell)

Lessons in Experimental and Practical Geometry. By H. S. HALL and F. H. STEVENS. Pp. vi, 94, iii. 1s. 6d. 1905. (Macmillan.)

An Introduction to Projective Geometry and its Applications: An Analytic and Synthetic Treatment. By A. EMCH. Pp. vii, 267. $2.50. 1905. (Wiley; Chapman & Hall.)

Formelsammlung und Repetitorium der Mathematik. By O. Th. Burklen. Pp. 227. 80 pf. 1905. (Goschen.)

Schattenkonstruktionen. By J. VONDERLINN. Pp. 118. 80 pf. 1905. (Göschen.)

Darstellende Geometrie. Vol. I. *Elemente Ebenflachige Gebilde.* By R. HAUSSNER. Pp. 207. 80 pf. 1905. (Goschen.)

Gazeta Matematica. Edited by I. IONESCU. Jan., 1905.

Le Calcul Simplifié. By MAURICE D'OCAGNE. P. viii, 225. 5 frs. 1905. (Gauthier-Villars.)

The Annals of Mathematics. Edited by OSMOND STONE and Others. Second Series. Vol. 6, No. 2. Jan., 1905. (Longmans, Green.)

The Solutions of Differential Equations of the First Order as Functions of their Initial Values. G. A. BLISS. *On the Conformal Representation of certain Isosceles Triangles upon the Upper Half-Plane.* L. WAYLAND DOWLING. *Remarks on a Proof that a Continuous Function is uniformly continuous.* N. J. LENNES.

The American Journal of Mathematics. Edited by F. MORLEY. Vol. XXVII., No. 1. 1905. (Kegan Paul.)

Some properties of a generalised Hypergeometric Function. F. H. JACKSON. *The Relation between real and complex Groups with respect to their Structure and Continuity.* S. E. SLOCUM. *Determination of all the characteristic Sub-Groups of any Abelian Group.* G. A MILLER. *Collineations whose characteristic Determinants have linear Elementary Divisors. Application to Quadratic Forms.* A. B. COBLE. *On certain Elliptic Modular Functions of Square Rank* J. A. MILLER. *Minors of Axi-Symmetric Determinants.* E. J. NANSON. *On the Forms of Sextic Scrolls having a Rectilinear Directrix.* V. SNYDER.

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THE DEVELOPMENT OF GEOMETRICAL METHODS.

XIII.

AMONG those who have contributed to the development of Infinitesimal Geometry, Sophus Lie is distinguished by several important discoveries which place him in the first rank. He was not one of those who showed from childhood a very marked aptitude, and when he was leaving the University of Christiania in 1865, he was still hesitating between Philology and Mathematics. It was Plucker's works which made him fully conscious for the first time of his vocation. He published, in 1869, his first paper on the interpretation of imaginaries in Geometry, and by 1870 he was in possession of the ideas which guided the whole of his career.

At that time I often had the pleasure of meeting him, and of talking to him in Paris, which he was visiting with his friend, F. Klein. A course of lectures given by M. Sylow revealed to him the full importance of the theory of substitutions. The two friends studied this theory in C. Jordan's great treatise. They were fully conscious of the important rôle that it was called upon to play in many branches of mathematical science in which it had not been as yet applied. They were both fortunate enough to succeed in giving to mathematical research the direction which appeared to them to be the most fruitful.

In 1870 Sophus Lie presented to the *Académie des Sciences* of Paris an extremely interesting discovery. Nothing is less like a sphere than a straight line, and yet Lie had imagined a singular transformation which made a sphere correspond to a straight line, and therefore enabled him to connect every theorem relative to straight lines with a theorem relative to spheres, and *vice versâ.* In this very remarkable method of transformation each property relative to the lines of curvature of a surface gives a theorem relative to the asymptotic lines of the transformed surface. The name of Lie will remain connected with these relations, which connect one to the other the straight line and the sphere, the two essential and fundamental elements of geometrical research. He developed them in a Memoir full of new ideas which appeared in 1872.

The investigations which followed this brilliant *début* fully confirmed the hopes which he had aroused. Plücker's conception of the generation of space by straight lines, by curves, or by surfaces arbitrarily chosen, opens to the theory of algebraical forms a field which has not been explored, and to which Clebsch has barely begun to assign the limits. But by the study of Infinitesimal Geometry the value of this conception was fully shown by Sophus Lie. The great Norwegian geometer first found in it the idea of congruences and complexes of curves and then that of *contact transformations*, the first germ of which he had found (for the case of the plane) in Plücker. The study of these transformations led him, simultaneously with Mayer, to perfect the methods of integration which Jacobi had invented for partial differential equations of the first order; in particular, it throws a dazzling light on the most difficult and obscure parts of the theories relative to partial differential equations of a higher order. It enabled Lie, in particular, to integrate all the cases in which Monge's method of characteristics is fully applicable to equations of the second order with two independent variables.

By continuing the study of these special transformations, Lie was led to construct progressively his masterly theory of continuous groups of transformations, and to bring to light the important rôle played by the idea of groups in geometry. Among the essential elements of these researches I may point out the infinitesimal transformations, the idea of which was his exclusively.

Three great treatises published under his direction by skilful and devoted collaborateurs contain the essential part of his work, and its applications to the theory of integration, to that of complex unities, and to non-Euclidean Geometry.

XIV.

Thus by an indirect path I have reached the non-Euclidean Geometry which is daily assuming greater importance in the researches of geometers. If I were the only speaker who is to address you on Geometry, I should be delighted to recall to your minds everything that has been done in this subject from the days of Euclid or, at any rate, from the time of Legendre up to the present day. Discussed in turn by the greatest geometers of last century, the question has progressively enlarged its borders. It began with the celebrated postulate of parallels; it ends with the geometrical axioms as a whole.

Euclid's Elements, which have resisted the wear and tear of so many centuries, at least will have the honour before they disappear of inspiring a long series of admirably connected treatises, which will contribute in the most efficacious manner to the progress of Mathematics. And at the same time they will provide philosophers with a sound and well-defined starting point for the study of the origin and formation of our knowledge. I am sure that my distinguished collaborateur will not forget among the problems of the present day that which perhaps is the most important, and to which he has devoted himself with so much success. To him I leave the task of developing it with all the fulness it assuredly merits.

I have just spoken of the elements of Geometry. We must not forget the expansion they have undergone in the last five hundred years. The theory of polyhedra has been enriched by the beautiful discoveries of Poinsot on starred polyhedra, and by those of Mobius on single-faced polyhedra. Methods of transformation have widened the exposition of those elements. It may be said that the First Book now contains the ideas of translation and symmetry, that the second contains the theory of rotation and displacement, and that the third is based on inversion and the theory of homothetic figures.

But it must be recognised that it is owing to Analysis that the *Elements* have been enriched by their finest propositions. To the highest Analysis we owe the inscription of regular polygons of 17 sides, and analogous polygons. To it we owe the long sought-for proofs of the impossibility of the squaring of the circle, and the impossibility of certain constructions with the ruler and the compass. Finally, it is to Analysis that we owe the first rigorous proofs of the maximum and minimum properties of the sphere. Geometry will now intervene where it has been preceded by Analysis.

What will the Elements of Geometry be like at the end of the century which has just begun? Will there be one single elementary Geometry? Perhaps America, with its schools freed from the fetters of courses and traditions, will give us the best solution of this important and difficult question. Von Staudt has sometimes been called *the Euclid of the Nineteenth Century*; I should prefer to call him *the Euclid of Projective Geometry*; but will this Geometry, however interesting it may be, furnish us with the sole basis for the Elements of the future?

XV.

It is now time to bring to a close a summary that has already lasted too long, and yet there is a large number of interesting investigations which I have been compelled to pass over in silence. I should have liked to have told you of Geometries of any number of dimensions, the conception of which goes back to the early days of Algebra, but the systematic study of which was only begun some 60 years ago by Cayley and Cauchy. This subject of research has found favour in your country, and I need not remind you that your illustrious President,* after having shewn himself a worthy successor of Laplace and Le Verrier, in a space which he with us considers as enjoying three dimensions, has not disdained to publish in the *American Journal of Mathematics* considerations of the greatest interest on the geometries of n dimensions. There is one objection, and one only, which may be levelled at researches of this kind, and it has already been raised by Poisson: the absence of any real basis, of any *substratum* which will enable us to present visibly, and as it were palpably, the real results obtained. The extension of the methods of Descriptive Geometry, and especially the use of the conceptions of Plucker on the generation of space will contribute to deprive this objection of much of its validity.

I should also like to have spoken to you of the method of equipollences, of which we find the germ in the posthumous works of Gauss, of Hamilton's Quaternions, of the methods of Grassmann, and in general of systems of complex unities, of *Analysis Situs*, so intimately connected with the theory of functions, of the so-called Kinematic Geometry, of the theory of Abaci, of Geometrography, of the applications of Geometry to Natural Philosophy and to the Arts. But I am afraid lest some analyst, as happened once before, may accuse Geometry of wishing to annex every field of research.

My admiration for Analysis, now so fruitful and so powerful, prevents my even thinking of such a step. But if any such reproach can be formulated in these days, it should not be addressed to Geometry, but, I think, to Analysis. The circle in which at the beginning of the nineteenth century the study of Mathematics appeared to be enclosed has been broken on all sides. The old problems are presented to us in a new form; new problems arise, and are investigated by legions of workers. The number of those who study Pure Geometry has become prodigiously restricted. Here is a danger against which we must guard. Let us not forget that if Analysis has acquired means of investigation which sometimes play it false, it largely owes them to the conceptions introduced by Geometers. Geometry must not remain buried, as it were, under its triumph. In its school we have learned, and our successors will have to learn, that we must never blindly trust to methods that are too general; that we must look at questions in themselves, and find in the particular conditions of each problem, either a direct road to a simple solution, or the means of applying in an appropriate manner the general processes which it is the duty of science as a whole to accumulate. As Chasles writes in the preface to his *Aperçu Historique*: "The doctrines of Pure Geometry often offer, and in a host of questions, that simple and natural method which, penetrating to the origin of truths, lays bare the mysterious chain which unites them one to the other, and makes them individually known in the most complete and luminous manner."

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* Professor Simon Newcomb.

Let us then cultivate Geometry, which has advantages all its own, and has no desire to outstrip its rival at every point. Besides, even if we tried to neglect Geometry, it would not be long before it found, in the applications of Mathematics, as it found once before, an opportunity of regeneration and development. It is like the giant Antaeus, who grew stronger every time he touched his Mother Earth. G. DARBOUX.

PROF. BRYAN'S "MEAN RATE OF INCREASE."

A MECHANICAL ILLUSTRATION.

THE Mean Rate of Increase which was investigated by Prof. Bryan in No. 48 of the *Mathematical Gazette* may evidently be thought of as the inclination of a straight line to an axis. The object of the following note is to point out an example in which this inclination naturally occurs.

Consider a rigid straight rod suspended by vertical elastic wires P_0Q_0, P_1Q_1, etc., from points P_0, P_1, etc., which are not in the same horizontal plane but at depths y_0, y_1, etc., below a certain horizontal plane. Let the number of wires be $2m+1$ and let P_0Q_0 be the middle one. The following definitions are also needed:

k = Natural length of each wire.

l = Mean length of the wires.

a = Mean value of the y's.

ϵ = Hooke's modulus of elasticity of wires.

M = Mass of the rigid uniform rod.

c = Distance between consecutive Q's.

θ = Inclination (supposed small) of rod to horizon.

The equations of virtual work are found by varying l and θ in the expression

$$(\epsilon/2k)\sum_{r=-m}^{r=m}(l+a+rc\theta-y_r-k)^2-Mg(l+a) \quad\text{.........(1)}$$

The equations are

$$(\epsilon/k)\Sigma(l+a-k+rc\theta-y_r)=Mg$$

or

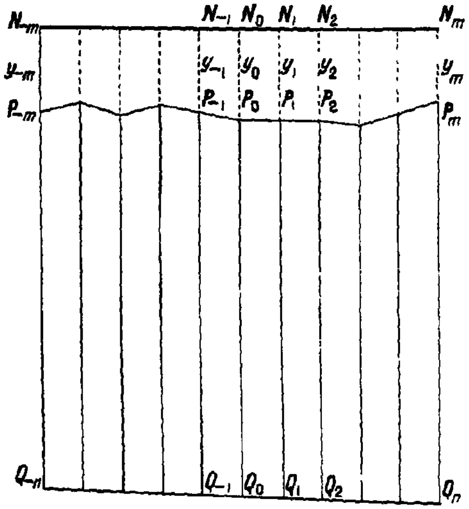
$$l-k=\frac{k}{\epsilon}\frac{M}{2m+1}\cdot g, \quad\text{.........................(2)}$$

and

$$\Sigma\, rc(l+a-k+rc\theta-y_r)=0$$

or

$$c\,.\,\theta=\frac{\sum_{-m}^{m} ry_r}{m(m+1)(2m+1)/3}. \quad\text{..................... (3)}$$



Equations (2) and (3) could be found directly by resolving vertically and by taking moments about the middle point of the rod.

Comparison with Prof. Bryan's notation shews that the "mean rate of increase" of the y's is the θ which is the inclination of the line of Q's to the horizontal. It should be noticed that the result is independent of the modulus of elasticity of the wires.

As a numerical example we may consider a girder suspended by equal elastic wires from points 1 metre apart and at depths, in *mm.*, below a horizontal plane

7·4, 6·7, 7·4, 7·3, 7·6, 8·6, 8·7, 8·5, 9·4, 8·8, 7·8.

Prof. Bryan's calculation shews that the inclination of the girder will be $1{\cdot}81 \times 10^{-4}$ in circular measure, and that the difference of level between the two ends of the girder will be 1·81 mm.

The definition of the mean rate of increase can easily be extended to the case when y is a continuous function of x between limits $x = \pm h$.

On applying the conditions that a and b are such that

$$\int_{-h}^{h} (y-a-bx)^2 dx$$

is a minimum, it appears that the mean rate of increase is

$$b = 3\int_{-h}^{h} yx\,dx/2h^3 \quad\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

$$= \frac{y_h - y_{-h}}{2h} - \tfrac{3}{4}\int_{-h}^{h} \left\{\frac{dy}{dx} - \frac{y_h - y_{-h}}{2h}\right\}\left(\frac{x}{h}\right)^2 dx. \quad\ldots\ldots\ldots(5)$$

The mechanical illustration is extended to this case by supposing that the number of wires becomes infinite. Unfortunately this is not the same as substituting an elastic membrane for the wires. A fabric in which all the threads of the woof are vertical is a better representation, but departs from the ideal simplicity which is associated with continuity.

F. J. W. WHIPPLE.

MATHEMATICAL NOTES.

165. [L¹. 3. a.] *On the equation to the asymptotes of the general conic.*

The equation to the conic being

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the equation, referred to parallel axes through the centre, is

$$ax^2 + 2hxy + by^2 + \frac{\Delta}{C} = 0.$$

Hence, with the centre as origin, the equation to the asymptotes is

$$ax^2 + 2hxy + by^2 = 0,$$

which may be written

$$\left(ax^2 + 2hxy + by^2 + \frac{\Delta}{C}\right) - \frac{\Delta}{C} = 0.$$

Changing the axes back to the original axes, this equation is transformed to

$$S - \frac{\Delta}{C} = 0, \text{ or } CS = \Delta,$$

the well-known equation to the asymptotes of the conic $S = 0$.

R. H. PINKERTON.

166. [X. 4. a.] *A glimpse of the Obvious.*

A simple diagram model for illustrating the significance of the arbitrary constant introduced by integration.

With the usual notation the increment of area $\left.\begin{matrix}PQM'M\\ \Delta A\end{matrix}\right\}$ in the figure is evidently intermediate between $\left.\begin{matrix}PKM'M\\ y\Delta x\end{matrix}\right\}$ and $\left.\begin{matrix}QLMM'\\ (y+\Delta y)\Delta x\end{matrix}\right\}$ and hence in the limit we have

$$\frac{dA}{dx}=y=f(x) \text{ say;}$$

or, expressing the same fact in the integral language,

$$A=\int (x)dx.$$

Now the second equation can assert neither more or less than the first. The first equation tells us the rate at which the area is growing, and is therefore independent of the point from which we commence to measure the area. Hence the second equation can only give an expression for the area measured from an undetermined starting point: and hence the reasonableness of the introduction of an arbitrary constant on integration is manifest.

To determine the area comprised between the curve $y=f(x)$, the axis of x and the ordinates at $x=a$ and at $x=b$ we have,

$$\left[\text{supposing } \int f(x)dx=F(x)+C\right]$$

area from the undetermined starting point up to $x=a$ is $F(a)+C$,
,, ,, $x=b$ is $F(b)+C$;

wherefore the required area is $F(b)-F(a)$.

To illustrate these facts the curve $\left(\text{for instance } y=\frac{1}{1+x^2}\right)$ and the coordinate axes are drawn in black on a sheet of paper, and the corresponding integrated equation (here $y=\tan^{-1}x$) and the axis of y are drawn in red to an identical x scale on a sheet of tracing linen.

Placing the tracing cloth over the other figure so that the y axes are superposed it is easily seen that the ordinate of the red curve at any point gives the "area of the black curve" up to the corresponding value of x, from a starting point determined by the intersection of the red curve with the axis of x; and that this statement holds good however the tracing cloth is moved up or down, provided the y axes are kept superposed. The meaning of the expression above for the area between the curve, the axis of x and two given ordinates may be illustrated in like manner. It may be worth while to give a still simpler illustration by using as the black line $y=m$ and as the red line $y=mx$.

C. S. JACKSON.

167. [K. 13. a.] *Remarque minuscule.*

Given in a plane or in space two segments AB, $A'B'$; then if AA', BB' are divided in C and D so that $\frac{AC}{CA'}=\frac{AB}{A'B'}=\frac{BD}{DB'}$ the straight line CD is parallel to the bisector of the segments AB, $A'B'$. If A coincides with A' we have the fundamental property of the bisector.

G. DARBOUX.

168. [P. 1. f.] *A geometrical construction.*

Lines on a sphere are projected from any point O on to the polar plane of O. Give a simple geometrical construction for the angle between two small circles whose projections are two given straight lines.

Solution.

Let the polar plane of O meet the sphere in a circle j whose centre is C; let VA and VB be the given straight lines. Let CV meet in N the polar

NAB of V with respect to j. Let a circle with centre N orthogonal to j meet CV in W. AWB is the angle required.
H. HILTON.

169. [**K. 21. b.**] *An approximate construction for the trisection of an angle.*

Let ABC be the angle to be trisected. when
Draw $YHMO$ parallel to the bisector of ABC intersecting AB in H. ABX
Draw BY perpendicular to YO.
Find L on AB and M on YO, equidistant from H, so that $LM=2BY$.
With centre B, radius BL, describe a circle cutting YO in O.
Bisect the angle MBO by BX'.
Bisect the angle ABX' by BR cutting YO in Z.
Draw ZT perpendicular to BR and equal to YB.
Draw TX parallel to BR.
This construction has reached me from a correspondent.
He thinks that BX is one of the lines of trisection. The error is $\frac{1}{3}XBX'$.

3rd April, 1905. W. H. H. HUDSON.

REVIEWS.

Lectures on the Calculus of Variations. By OSKAR BOLZA. Pp. xv + 271. The University of Chicago Press, 1904. [Decennial Publications of the University of Chicago, Second Series, volume 14.]

The methods of the Calculus of Variations have been almost entirely recast during the last 33 years: in this subject, as in most other branches of analysis, the first advances were deficient in rigour. The critical revision of the foundations was undertaken first by Weierstrass, who gave ten courses of lectures on the subject (1865-1889); and, from statements in Professor Bolza's book (§§ 24, 28), it appears that Weierstrass's chief discoveries were given in his courses of 1872 and 1879. In the former, the parametric method was used;* in the latter, it was shewn that the three conditions previously known to be *necessary* for a minimum must be supplemented by a fourth condition; and these four conditions were proved to be *sufficient*, a conclusion which could not be obtained from the older methods (see Bolza, § 17). Professor Bolza had the privilege of hearing the first exposition of these discoveries, which mark an epoch in the history of the subject.

For some 16 years Weierstrass's work remained generally unknown in this country; but in 1895-6, Professor Forsyth inaugurated a course at Cambridge on the subject. Since that date, Kneser has published his text-book (in 1900), and convenient summaries are now available;† besides which, a number of articles have appeared in the American Mathematical Journals. Last year, too, Professors Hancock and Bolza published courses of lectures, one of which is the subject of this review.

According to Professor Bolza's preface, his book originated in lectures delivered at the 1901 summer meeting of the American Mathematical Society; although the work has been somewhat modified to make it suitable for a text-book. Turning now to details, the theory of minimising the integral

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* It does not appear to be certain whether Weierstrass had used this method in any earlier courses; according to the table of lectures given in Bd. 3 of Weierstrass's works, the course in 1872 was his fourth on the subject.

† *Encyclopaedia Brit.*, vol. 33 (by Professor Love); and *Encyclo. d. Math. Wiss.*, Bd. 2 (by Kneser, Zermelo and Hahn).

$$(1) \qquad \int_{x_0}^{x_1} F(x, y, y')dx, \qquad (y' = dy/dx).$$

is contained in chapters 1-3; and here I would direct special attention to the careful tabulation (pp. 101, 102) of the *sufficient* conditions in contrast to the *necessary* conditions.* Chapters 4-6 give the corresponding theory, in Weierstrass's parametric form, for the integral

$$(2) \qquad \int^{t_1} F(x, y, x', y')dt, \qquad (x' = dx/dt,\ y' = dy/dt).$$

Chapter 5 contains Kneser's theory, which depends on an extension of Gauss's theorems on geodesics; chapter 6 the theory of isoperimetric (or *relative*) problems. Chapter 7 (the last) gives an account of Hilbert's theorem on the existence of solutions of minimum problems, under certain restrictions on the function F.

That Professor Bolza is a pupil of Weierstrass's may be seen from every page of this work; but he is not merely content to reproduce Weierstrass's results, he revises them where necessary.† But personally I am inclined to regret that Professor Bolza did not see his way to make more use of Hilbert's invariant-property (§§ 21, 37) that the integral

$$\int_{x_0}^{x_1} [F(x, y, p) + (y' - p)F_p(x, y, p)]dx, \qquad (y' = dy/dx).$$

is independent of the path;‡ for this property leads to very attractive proofs of the leading theorems of the subject. In particular the complicated analysis belonging to the theory of the second variation may be avoided; for details I refer to Hedrick's article, quoted above.

It may seem strange to English readers, accustomed to the text-books of Todhunter and Williamson, that Bolza restricts the theory to integrals of the types (1), (2), which contain no derivates beyond the first. There are two reasons for this; first, hardly any practical problems contain higher derivates; secondly, the theory for such cases is not so complete. Perhaps, too, some may be sceptical about the weakness of the older methods; I therefore refer briefly to one objection. Up to 1879, it was invariably assumed that a small displacement of a curve involved a small change in shape also; but this assumption really limits the variations permissible.* Thus the old theory admits, as a variation of the axis of x, a curve of small ordinates, *only if the slope and curvature are also small*; but it excludes the curve obtained by drawing a succession of small semi-circles on alternate sides of the axis.

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* Compare E. R. Hedrick, *Bull. Amer. Math. Soc.*, vol. 9, 1902, p. 23. Note that necessary *and* sufficient conditions are known for the integral (2), but not for (1).

† I quote from a review by Stäckel in the *Archiv der Math. u. Phys.*, Bd. 8, 1904, p. 259: "Nicht der ist ein treuer Schuler von Weierstrass, wer kritiklos hinnimmt, was dieser gelehrt hat, sondern wer bemüht ist, nach seinen Kräften die Teile der mathematischen Wissenschaft zu fördern, die dem Meister am Herzen lagen."

‡ Here $p(x, y)$ is a one-valued function of x, y, such that $y' = p(x, y)$ is a first integral of the differential equation of the problem arising from the integral (1).

After becoming acquainted with the Weierstrassian parametric method, one is tempted to regard the methods based on the integral (1) as out-of-date and incomplete. But, as Bolza remarks in a footnote on p. 115, the two methods really apply to *different* problems; the parametric being specially adapted to geometrical problems, and the other to analytical problems, where y is to be found as a function of x. I may illustrate this contrast by Weierstrass's theorem (p. 142) that *the integral* (2) *has no minimum if F is rational in x', y'*; but, on the contrary, *the integral* (1) *may have a minimum, even though F is rational in y', provided y is restricted to be a one-valued function of x.* For example, the integral $\int_0^1 (dy/dx)^2 dx$ has a minimum given by $y=x$, if the terminal conditions are $y=0$ for $x=0$, and $y=1$ for $x=1$. But the equivalent parametric integral $\int (y'^2/x')dt$, taken from (0, 0) to (1, 1), has no minimum.†

The last example recalls a more celebrated one which might have been mentioned by Bolza in connection with his account of Hilbert's existence-theorem; I refer to the integral $\int_0^1 x^2 (dy/dx)^2 dx$, with the same end-conditions. As Weierstrass shewed in 1870, the differential equation derived from this integral admits no continuous solution with

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* Expressed analytically, the old assumption amounts to taking variations of the type $\epsilon\phi(x)$, where ϵ is small and $\phi(x)$ is arbitrary, *but does not contain ϵ*. This would exclude $\epsilon \sin(x/\epsilon)$. Variations such as $\epsilon\phi(x)$ are now called *weak variations.*

† For, put $y=x+\xi$, where ξ is a continuous one-valued function of x, which is zero for $x=0$ and $x=1$; then, after integration, we find that the first integral becomes

$$\int_0^1 (dy/dx)^2 dx = 1 + \int_0^1 (d\xi/dx)^2 dx,$$

shewing that $\xi=0$ makes the integral a minimum. But in the parametric form the same transformation gives

$$\int_0^1 (y'^2/x')dt = 1 + \int_0^1 (\xi'^2/x')dt;$$

and, as x' may now be *negative*, the minimum property no longer holds. In particular, consider the "curve" of zig-zag straight lines given by

$$\left.\begin{array}{ll} x=\frac{n+1}{n-1}t,\ y=0, & \text{from } t=0 \text{ to } \frac{n-1}{2n} \\ x=1-t,\ y=nt-\frac{1}{2}(n-1), & \text{from } t=\frac{n-1}{2n} \text{ to } \frac{n+1}{2n} \\ x=\frac{n+1}{n-1}t-\frac{2}{n-1},\ y=1, & \text{from } t=\frac{n+1}{2n} \text{ to } 1 \end{array}\right\} n>1.$$

This curve makes the value of the parametric integral $-n$; and hence, as n is not limited, the integral can have no minimum. Of course, on this curve, y is *not* a one-valued function of x.

the given end-conditions; further, by taking[*] $y\theta = \arctan(x \tan\theta)$, where θ is slightly less than $\frac{1}{2}\pi$, the integral may be made arbitrarily small, leading to the inference that it has no minimum (because $y = \text{const.}$ does not satisfy the end-values). This example shews that Hilbert's existence-theorem for the integral (1) fails if $\partial^2 F/\partial y'^2$ is zero, even once, in the range of the integral. From a historical standpoint, this example is also of interest, for it first exposed the fallacy on which "Dirichlet's principle" was based by Lord Kelvin and Dirichlet;[†] and, in a sense, it may be said that this example led to Hilbert's theory, as he seems to have been guided by the hope of resuscitating the Kelvin-Dirichlet method.[‡]

In conclusion, it may be said that an excellent feature of Bolza's book is the systematic use of special problems to illustrate general theorems; but the addition of an index to these problems (as in Kneser's book) would have rendered the book more useful to one who needs to look up all the details of a particular problem. I heartily commend the book to any readers who wish for a connected account of modern results in the Calculus of Variations. T. J. I'A. B.

Rational Geometry, based on Hilbert's Foundations. By G. B. Halsted. (New York: John Wiley & Sons, 1904, pp. 285.)

Mr. Halsted has added one more to the list of the classics which he has made accessible. The present volume is more than a translation. The earlier part is an adaptation of Hilbert's *Grundlagen der Geometrie* (also translated into French by M. Laugel and into English by Mr. E. T. Townsend).

It may be said at once, and would doubtless be admitted by Mr. Halsted, that this book has little or nothing to do with school work. School geometry must be based upon intuition, which Hilbert and Mr. Halsted exclude. At the same time it is well that some schoolmasters

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[*] I note, incidentally, that this value of y makes $\int_0^1 (x^2 + \cot^2\theta)(dy/dx)^2 dx$ a minimum, if y is restricted to be a one-valued function of x. In fact, if we write

$$y = \frac{1}{\theta} \arctan(x \tan\theta) + \xi,$$

ξ being subject to the same conditions as before, we find

$$\int_0^1 (x^2 + \cot^2\theta)(dy/dx)^2 dx = \frac{1}{\theta}\cot\theta + \int_0^1 (x^2 + \cot^2\theta)(d\xi/dx)^2 dx.$$

[†] Their argument is that the integral

$$\iiint \left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)\right] dx\,dy\,dz,$$

taken through a volume on the boundary of which V is to have assigned values, *must* have a minimum because the subject of integration is never negative. Granting this assumption, the existence of a solution of the potential equation $\Delta V = 0$, with given boundary-values can be deduced. But the same argument applies to Weierstrass's example, for which the corresponding conclusions are *wrong*; because the integral has no minimum, neither has the differential equation a solution (with the given end-conditions).

[‡] In his first account of the theory (*Jahresbericht d. D. Math. Verein.*, Bd. 8, 1900, p. 184), Hilbert says: "Das folgende ist ein Versuch der Wiederbelebung des Dirichlet'schen Princips."

should be acquainted with the line of thought here developed. We could have wished that Mr. Halsted's plan had included a commentary; the matter is set out with Euclidean severity, and the reader has to work hard.

Hilbert's first quarrel with the traditional geometry is about congruence. When is one finite straight line AB (which Mr. Halsted calls "sect") to be considered congruent with another sect XY? Euclid answers: When AB can be moved so as to coincide with XY. But, of course, AB must not alter in length while it is being moved. Now, what does this mean? It means that if $A'B'$ is any portion of AB during the translation, then $A'B'$ is to be congruent with AB. But what does congruent mean? This is just what we are trying to define. And we are arguing in a circle. "To try to prove the congruence assumptions and theorems with the help of the motion idea is false and fallacious, since the intuition of rigid motion involves, contains, and uses the congruence idea." To define congruence of sects and angles without motion, Hilbert resorts to a set of assumptions. It is curious that he is forced to *assume* Euclid I. 4 as far as the equality of the base angles: he can then prove the equality of the bases.

He is unable to prove the congruence of triangles which have congruent two pairs of angles, and a pair of sides not included (Euclid I. 26, Case 2). This appears to lead to a second ambiguous case, as would happen in the surface of a sphere.

We learn that "no assumption about parallels is necessary for the establishment of the facts of congruence or motion." Playfair's axiom is adopted.

The chapter on "constructions" is interesting. Apparently all figures whose existence can be deduced from the assumption admit of construction with ruler and "sect-carrier," *e.g.* trisection of sect is possible, and trisection of angle impossible. Hilbert shows that there are constructions possible with ruler and compass which are not possible with ruler and sect-carrier.

Coming to area, we find that the rejection of intuition leads us along a thorny path. For reasons which we dimly apprehend, Mr. Halsted refuses to associate numbers with sects (he never gives a numerical measure of the length of a line), and will have nothing to do with limits. (Hilbert is more generous.) Two polygons are defined as equivalent if they can be cut into a finite number of triangles congruent in pairs. After proving the equivalence of parallelograms in the same base and between the same parallels, Hilbert is seized with misgivings —perhaps *all* polygons are equivalent. These doubts are resolved, and the section ends with the demonstration that "a polygon lying wholly within another polygon must always be of lesser content than the latter."

A similar procedure is necessary in dealing with the volumes of polyhedra. The work on curved lines and curved surfaces appears to be due to Mr. Halsted. We cannot think that it attains the standard of rigour claimed for it. The circumference of a circle is stated to be π diameters, but we do not find any explanation of the meaning of (diameter $x\pi$). The idea of a limit must have crept in. The area of a sector of a circle is *defined* as the product of its arc by half the radius.

"Product" is defined satisfactorily, and Mr. Halsted has a right to define "area of sector" as he likes; but this definition gives us no clue to what would be meant by the area of an ellipse, say. No general definition is given of the area of a curved surface; but in § 453 we are told that the lateral area of a right circular cone is the same as that of a sector of a circle with the slant height as radius and an arc equal in length to the length of the cone's base. Is this intuition? or a latent definition? or a limit theorem? To unroll a conical surface into a plane would seem to be as objectionable as to wave a straight line. Again, the area of a sphere is *defined* as the quotient of its volume by one-third its radius. Has the area of a sphere so defined any relation to the area of a cone?

The volume of a sphere (or other curved surface) is virtually defined by Cavalieri's assumption: "If the two sections made in two solids between two parallel planes by any parallel plane are of equal area, then the solids are of equal volume." A sphere is then compared in an ingenious way with a tetrahedron.

The statement on page 275, that no one has examined the primeness of $2^{2^m}+1$ for values of m beyond 7, must be modified in the light of recent work by Colonel Cunningham and Mr. A. E. Western.

Our parting impression is that Mr. Halsted has produced an interesting book, but has not entirely justified his prefatory remark, "Geometry at last made vigorous is also thereby made more simple." C. G.

Lessons in Experimental and Practical Geometry. By H. S. HALL and F. H. STEVENS 1905. (Macmillan.)

Within the lines laid down for it in the preface this little book may be cordially recommended. The illustrations are good, the letterpress clear and concise. The matter and the methods are purposely restricted, and the point of view is throughout experimental. The objects of the book are best qualified in the words of the preface: "In the preface to our *School Geometry* it is suggested that a suitable introduction to that book would consist of 'Easy Exercises in Drawing to illustrate Definitions; Measurement of Lines and Angles; The Use of Compasses and Protractor; Problems on the Construction of Triangles and Quadrilaterals: these problems to be informally explained and the results verified by measurement.' It is the purpose of these *Lessons* to supply such a practical course." G. H. YOUNG.

Theory of Determinants. By R. F. SCOTT, M.A. Second Edition. Revised by G. B. MATHEWS, M.A., F.R.S. Pp. 288. 9s. (Cambridge University Press.)

In this new edition an effort has been made to make the book more suitable for beginners. To this end an introductory chapter is given and difficult notations are omitted from the more elementary parts. Two new chapters are introduced, one on elementary divisors and the other on determinants of infinite order. The chapter on quadratic forms has been enlarged to include the elements of bilinear forms, and occasionally one finds a new theorem introduced into some of the other chapters. Theorems which might have found a place are Schwein's theorem, of which Laplace's expansion theorem is a particular case, the law of extensible minors, the law of complementaries, etc. By means of these many of the proofs might be shortened. Slight modifications occur here and there in the wording, due in part to the introduction of new matter and rearrangement of the old, and in part to clearer statements and simplified proofs. The book on the whole is comparatively little modified, but where changes have been made they are in general improvements. The theorem of article 17, chapter vi., attributed to Netto, was not new with him, and therefore should hardly bear his name. It is found in Reiss, which was published in 1867. W. H. METZLER.

Elementary Algebra. Part II. By W. M. BAKER and A. A. BOURNE of Cheltenham College. 1904. (G. Bell and Sons.)

The strong point of this book is the large and varied assortment of examples, many of which are interesting and instructive. Here and there too the teacher will find an ingenious suggestion, as for instance the neat geometrical illustration (p. 312) of the result of adding equal quantities to each term of a ratio.

But these valuable features are marred by what must be termed the slovenly inaccuracy, often of expression and sometimes of substance, of the text. To prove the justice of this somewhat severe comment I might refer to the treatment of fractional indices, or of the binomial theorem for a negative or fractional index, or of partial fractions, or to such sentences as "draw *PQ* equal to *56 shillings.*"

A typical instance is offered by the attempt (§ 260) at a graphical solution of the problem: "A man mixes wine at 30*s.* a dozen with wine at 80*s.* a dozen. How many dozen of each kind must he take in order that a mixture of 60 dozen may be worth 50*s.* a dozen?"

A diagram is given in which price per dozen and number of dozens are plotted as coordinates and the solution begins "In the diagram *OA* is the graph of the 30*s.* wine."

The line *OA* (or in fact any line through the origin) has of course no more connection with 30*s.* wine than with wine at any other price.

As further evidence § 325 may be quoted verbatim (a trifling misprint excepted).

"Approximation for the r^{th} root of a number.

If $\sqrt[r]{N}=a+x$ where a is an approximation and consequently x is small, then a closer approximation is given by the formula

$$\sqrt[r]{N}=\frac{(r+1)N+(r-1)a^r}{(r-1)N+(r+1)a^r}a. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)\text{"}$$

[Special case of $r=3$, and numerical approximation to $\sqrt[3]{2}$ follow.]

"Proof of (1).

$$\frac{(r+1)N+(r-1)a^r}{(r-1)N+(r+1)a^r}a=\frac{(r+1)N+(r-1)[N^{\frac{1}{r}}-x]^r}{(r-1)N+(r+1)[N^{\frac{1}{r}}-x]^r}a$$

$$=\frac{2rN-r.(r-1)xN^{1\ \frac{1}{r}}}{2rN-r.(r+1)xN^{1-\frac{1}{r}}}a=\frac{2-(r-1)\dfrac{x}{N^{\frac{1}{r}}}}{2-(r+1)\dfrac{x}{N^{\frac{1}{r}}}}a$$

$$=\frac{2-(r-1)\frac{x}{a+x}}{2-(r+1)\frac{x}{a+x}}a=\frac{2a+(3-r)x}{2a+(1-r)x}a$$

$$=\left(1+\frac{2x}{2a+(1-r)x}\right)a=a+\frac{x}{1+\frac{1-r}{2a}x}=a+x-\frac{(1-r)x^2}{2a}-\text{etc.}$$

$$=a+x \text{ approximately."}$$

It will be observed that no hint or clue is given as to how the formula was discovered. It might be part of some system of revelation for all that a schoolboy could infer. Further the expansion is effected wrongly, some terms of the second order being omitted and some retained.

Surely a preferable method would be to say that we seek an approximation to $N^{\frac{1}{r}}$ closer than a in the form $\frac{1+py}{1+qy}a$ where $y=N-a^r$.

Expanding $N^{\frac{1}{r}}$ and $\frac{1+py}{1+qy}a$ in powers of y we find that if the series are to be identical as far as the terms in y^2 inclusive we must have $p=\frac{r+1}{2ra^r}$ $q=\frac{r-1}{2ra^r}$ whence the result follows at once. The error in defect proves to be $\frac{r^2-1}{12r^3}\left(\frac{y}{a^r}\right)^3 . a$ nearly, provided $\frac{y}{a^r}$ is < 1.

Of course the result is still more easily obtained as a simple geometrical application of Rolle's theorem.

Other quotations might be given, but enough has perhaps been said to show that among the recommendations of the Mathematical Association which the authors "follow to a great extent" (preface), that one which urges that great stress should be laid on fundamental principles is not so fortunate as to be included. C. S. JACKSON.

Einleitung in die Funktionentheorie, (I Abteilung), von O. STOLZ und I. A. GMEINER. 1904.

This volume, the fourteenth of Teubner's new mathematical series, is a continuation of the same authors' *Theoretische Arithmetik*:* the two books will when completed form a new and much enlarged edition of the well-known *Allgemeine Arithmetik* of Prof. Stolz, and supersede that excellent work.

The range of the present volume is more like that of Harkness and Morley's *Introduction to the Theory of Analytic Functions* than that of any other English book. But it begins farther on and ends sooner, and the intervening ground is naturally covered with much greater thoroughness. So far as English readers are concerned it is only too likely that this book will fall between two stools. It is altogether too solid and systematic for one who is only beginning the theory of functions. On the other hand it contains nothing which an expert in the subject ought not to know perfectly well already. Still for a student who has acquired a superficial knowledge of a number of different branches of the theory, one, let us say, who has just taken Part II. of the Mathematical Tripos, and is about to begin to study the subject seriously, it is difficult to imagine a type of book more profitable to read. The authors develop the theory in a thoroughly consistent and logical way; they do not jump from one theory of functions to another, from Cauchy to Riemann, and from Riemann to Weierstrass, but proceed on strictly Weierstrassian lines throughout. So much is this the case that, in spite of the wealth of matter which the book contains, differential coefficients are only alluded to once or twice, and that quite incidentally, while the word 'integral' does not occur once.

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* See *Math. Gazette*, vol. II., p. 312.

As in their former book, the authors include numerous examples. This custom, which seems, happily, to be spreading rapidly abroad, is no doubt a result of the influence of English books. But there is nothing 'English' about the examples themselves. The authors adopt the excellent method of including among the examples all obvious corollaries and extensions for which there is no room in the text; and, whenever any example presents the slightest difficulty, hints are given which remove it. There is not an example in the book which would puzzle a Cambridge mathematician, who knows anything about the subject, *for five minutes. And, although no doubt we sin in the opposite* direction, most people will probably think that in this respect the authors go too far; it would be better to exercise the ingenuity of the reader a little more. However, of their kind the examples are excellent, and include some of types which ought to be but are not in every English book upon the subject.

There is not very much in the way of detail to which it is worth while to call attention. There is a serious logical error in the very first line, which recurs on p. 114. A variable, whatever it may be, is not a *sign*: it is what the sign denotes. How to define a 'variable' in strictly logical terms is no doubt a by no means easy question. But it is no more a sign than Prof. Stolz is his signature. On p. 26 the authors have missed an excellent opportunity of giving a short account of the general notion of 'orders of infinity.' Heine's theorem concerning uniform continuity (pp. 52-55) may be deduced from a general theorem first explicitly stated by Dr. Baker. 'If every point P of a certain region (in space of any number of dimensions) possesses the property that it can be enclosed in a small region such that, if Q is any other point within it, a relation $(P-Q)$ holds, then the whole region can be divided into a *finite* number of regions, each containing *at least one* point P satisfying the relation $(P-Q)$ with respect to all points Q of the sub-region within which it lies.' This point occurs, for instance, in Goursat's proof of Cauchy's theorem; and to state and prove it, and some similar general theorems, early in every book on the theory of functions would save a good deal of unnecessary repetition later on. The treatment of 'fonctions à variation bornée' (pp. 57-60) is particularly simple and lucid. On p. 62 the authors are not very clear as to what they mean by the 'possibility of geometrical representation' of a function. The theorems proved on pp. 89-95 concerning multiple and repeated limits are very interesting, although they do not go very far. The sections on complex functions of a *real* variable and their geometrical representation (pp. 108-114) are interesting and might well be drawn upon by elementary English books, as might the examples (pp. 133-134). The brief account of 'Arithmetic Series' and Interpolation is an excellent feature (pp. 160-171). The last chapter, giving an account of the Weierstrassian theory of Power Series, is really admirable; about the theory of Multiple Power series (pp. 219-236) just enough is said to whet the appetite for more.

It is to be hoped that in the concluding volume the authors will have something to say concerning the fundamental relations between Cauchy's and Weierstrass's theories of functions. It is certain that the last word on this most interesting subject has not yet been said. Cauchy's theory shews that the assumptions of the Weierstrassian theory are redundant. For in Cauchy's theory we need only assume that a function has a derivative at every point; we need not even assume that this derivative is continuous, much less that higher derivatives exist. All these things can be *proved.* But in the Weierstrassian theory we assume far more than this, in another language it is true. Why should this be? What is there in the notion of the integral in virtue of which its introduction simplifies the assumptions of the theory so immensely? G. H. HARDY.

N. J. Lobatschefskij's Imaginäre Geometrie und Anwendung der Imaginären Geometrie auf einige Integrale. Herausgegeben von H. LIEBMANN. (Teubner, 1904; pp. xi, 188.)

These two memoirs are not of the same fundamental importance as some of Lobatschewsky's work, but they are very interesting none the less, and Herr Liebmann has performed a real service to mathematics by translating them from the Russian and editing them in an intelligible form. One would gather from the editor's elaborate notes and from some remarks in the Introduction, which have an almost pathetic ring, that, if ever a mathematician needed careful editing, then, if not Sylvester, it was Lobatschewsky. His style, at any rate in these memoirs, is obscure and condensed to the verge of unintelligibility, and the details of his analysis often inaccurate; the editor's task must indeed have been arduous and somewhat thankless.

In order to understand these memoirs it is essential to realise the peculiar point of view from which they were written. The author's earlier work upon the subject, written in the synthetic and geometrical spirit more usually associated with his name, had been imperfectly understood, and to some extent called in question, in part on the ground of supposed absurd analytical consequences. Nettled by these criticisms Lobatschewsky returns to the subject from a different side. Laying down *a priori* certain relations supposed to hold between the sides and angles of a right-angled triangle, he deduces a complete and consistent system of trigonometrical formulae which he shows to lead by an imaginary transformation to the ordinary formulae of spherical trigonometry. He then devotes himself to applications of the differential geometry of non-Euclidean space to analysis, and in particular to the evaluation of definite integrals, just as one might apply the elementary geometry of the Euclidean circle to the proof of the formula

$$\int_0^a \sqrt{a^2 - x^2}\,dx = \tfrac{1}{4}\pi a^2.$$

In each formula obtained by such quasi-geometrical methods, and verifiable by pure analysis, he sees a new confirmation of the logical possibility of his geometrical theory. This theory would not be called in question by anybody now, so that a good deal of his work is only of historical interest.

At the same time, many of Lobatschewsky's results are interesting in themselves. This is most true, perhaps, of his investigations of the volumes of certain figures in non-Euclidean space, cones, cylinders, and tetrahedra.

But some of the integrals which he evaluates are interesting too; such, for instance, as

$$\int_0^\pi \frac{\sin^n x\,dx}{(a + b\cos x)^{n+1}} = \frac{\sqrt{\pi}}{(a^2 - b^2)^{\frac{n+1}{2}}} \frac{\sqrt{\left(\frac{n+1}{2}\right)}}{\sqrt{\left(\frac{n}{2}+1\right)}},$$

$$\int_0^{\frac{1}{2}\pi} \log\left(\frac{1 + \cos\beta\cos x}{1 - \cos\beta\cos x}\right)\frac{dx}{1 - \cos^2\alpha\cos^2 x} = \frac{\pi}{\sin\alpha}\log\left(\frac{1 + \sin\alpha}{\sin\alpha + \sin\beta}\right),$$

$$\int_0^\infty \frac{x\,dx}{\cosh 2x - \cos 2\alpha} = \frac{\alpha\log 2 - L(\alpha)}{2\sin\alpha\cos\alpha},$$

where
$$L(x) = -\int_0^x \log\cos x\,dx,$$

and so on. G. H. Hardy.

Theorie der Elecktricität und des Magnetismus. Vol. II. By J. Classen. (Hamburg: G. J. Göschen.)

This is the second volume of Dr. Classen's book, the first volume of which deals with electricity. The present volume deals with magnetism and electromagnetic theory.

Dr. Classen is to be congratulated on the powerful, clear, and logical method which he has brought to bear on the subject. We have not space to give an account of the whole book in detail, but he points out in a satisfactory way the limitations and assumptions involved in several propositions generally accepted. In particular, in his treatment of magnetic hysteresis he avoids the fallacy which unfortunately has crept into so many books, of starting with an energy function, and by curiously distorted reasoning, showing that it leads to a loss of energy in going round a cycle. The facts show that there is no definite expression for the energy, and Dr. Classen is careful to point out that any expression for the change of energy which leads to loss in a cycle must be regarded as an assumption which requires direct experimental proof.

G. W. Walker.

Schatten-konstruktionen. By PROF. J. VONDERLINN. Pp. 118. Price 80 pf. (Leipzig, G. J. Göschen.)

This little book is of a somewhat technical nature. It is customary, in architectural diagrams, to suppose that all objects represented are illumined by a parallel beam of light inclined at an angle of 35° 15′ both to the horizon and the plane of the drawing; and the object of this treatise is to provide a series of rules for filling in the boundaries of the shadows. These boundaries are of two kinds: those analogous to the terminator of the crescent moon (Selbstschattengrenze), and those analogous to the edge of the earth's shadow on the moon during a partial eclipse (Schlagschattengrenze). The methods given for finding the boundaries in the two cases are clear, and the illustrations cover a wide range of possible cases—mouldings, screws, surfaces of revolution, and so on. The reader will find some interesting geometrical exercises, if he sets himself to give a proof of the constructions used. H. H.

Über die Anwendungen der darstellenden Geometrie, inbesondere über die Photogrammetrie. By FRIEDRICH SCHILLING. Pp. iv + 198. (Teubner.)

This is a series of three lectures read at Göttingen to schoolmasters during the Easter Vacation of 1904. The author's object is to suggest suitable subjects for exercise in geometrical drawing, and to show that a wide range of problems can be solved by elementary graphical methods. The illustrations are selected from fourteen different branches of science, but the construction of stereographic and orthogonal crystal diagrams, of drawings in perspective, etc., are dismissed in a few pages as being familiar to all, and but little more space is allotted to graphical statics, optical illusions, machinery, astronomy, and so on. In fact the only subject discussed in detail is photogrammetry, *i.e.* the art of constructing a map of a district or a plan of a building with the help of photographs taken from two different points of view. In the case of a building of which we know that certain lines are horizontal or vertical, and that certain angles are right angles, one photograph may suffice. The book is artistically printed and illustrated; it will be read with great pleasure by anyone interested in perspective and other applications of geometrical drawing. There is a short appendix on the use of the magic lantern in mathematical teaching. H. H.

Neue Beiträge zur Frage des math. und phys. Unterrichts an den höheren Schulen. Theil I. By Von F. KLEIN, E. RIECKE and others. Pp. 190. (Teubner.)

This is a collection of papers read to schoolmasters at Gottingen during the Easter Vacation of 1904. Several of them have been published before (see *Math. Gazette*, Dec. 1904, p. 113); to these are added five others. E. Riecke gives an interesting sketch of the development of theory and of the most recent researches in electricity. O. Behrendsen advocates an early introduction of mechanics and the principle of energy into the school science syllabus. J. Stark asks that a course in physics should inflict less strain on the memory, should include only simple experiments, and should lay stress on the applications of the subject. E. Bose gives a suitable course of instruction and list of apparatus for practice in experimental dexterity. K. Schwarzschild describes some fairly simple apparatus for carrying out rough astronomical measurements, more useful perhaps in an amateur's study than in the schoolroom. H. H.

A New Geometry for Junior Forms. By S. BARNARD and J. M. CHILD. Pp. 304. 1904. (Macmillan & Co.)

This book is based on the more elementary parts of *The New Geometry for Schools.* It is divided into an "introductory section" containing very simple exercises with ruler, compasses, protractor, etc.; a "practical section with complete proofs"; and a "theoretical section" covering the ground of Euclid, Books I. and III., 1-31. The middle section contains some valuable exercises in geometrical drawing, but may give the beginner very slovenly ideas as to what a "proof" should mean. H. H.

Methodisches Lehrbuch der Elementar-Mathematik. Erster Theil.

By Gustav Holzmüller. Pp. xi+319. 1904. (Teubner.)

England is not the only country which is afflicted with the Euclidean controversy. The preface to the fourth edition of this well-known text-book is devoted almost entirely to the theory of geometrical teaching. The author substitutes a discussion of fundamental notions and a series of exercises in geometrical drawing for formal definitions, postulates, and problems. These pave the way for theorems based on congruence and symmetry, and thus most of Euclid I-VI is covered in an order very different from that adopted in this country. The remainder of the book is more English in its methods, except that Arithmetic and Algebra are taught side by side; a pleasing feature made possible by the use of a metric system. Indices and logarithms are included in the course, but progressions are absent. Twenty pages are devoted to the elements of trigonometry, and fifty to mensuration and exercises in perspective drawing; the examples in the latter part are mostly those useful in crystallography. H. H.

An Introduction to Elementary Statics (treated graphically).

By R. Nettell, M.A. 2s. (London, Edward Arnold.)

This Introduction to Elementary Statics consists of a number of examples to be solved by drawing force diagrams and then measuring. The collection shows only too clearly the main drawback to a purely graphical treatment—that principles are lost sight of, when the method is applied to many similar examples. The author might have gone further afield in making choice of his examples, and we hardly think that he has fulfilled the proposal expressed in the preface of "choosing, as far as possible, examples connected with familiar objects to every schoolboy." A claw-hammer, a bell crank lever, a letter weighing machine, a skylight window, and other familiar objects suggest themselves to one who has glanced through the pages of this book. R. M. M.

Elementary Geometry of the Straight Line, Circle, and Plane Rectilineal Figures.

By Cecil Hawkins. Part II. Pp. 167-296. 2s. 1904. (Blackie.)

We fear that quite innocently on our part we may have done some slight injury to the circulation of Mr. Hawkins' *Geometry* by a remark made on p. 144. The copy sent us from the publishers contained no answers, and naturally we commented adversely on the publication of this excellent edition without the numerical results. We now learn that the book is published with and without answers, the price in both cases being the same. We sincerely hope that owing to this oversight on the part of the publishers we have not induced any teachers to pass over a book of which we have otherwise spoken in terms of praise.

Logarithmic Plotting Scales.

Aston & Mander, 61 Old Compton Street.

These are boxwood scales graduated with values of logarithms, log 10 or unity being represented by 10 inches. They are intended for use in graphical work illustrative of the leading properties of logarithms.

A straight line is so much more easily drawn than any other locus that it is often worth while to transform an equation so that the corresponding graph becomes a straight line.

Thus, if $v=u^n$ and we plot $x=\log u$, $y=\log v$, we have a "logarithmic" graph which is a straight line.

Again, if $v=ae^{n}$, and we put $\log_{10}e=m$, $\log_{10}a=b$, $x=u$, $y=\log_{10}v$, and plot x and y, we obtain a semi-logarithmic graph which is a straight line. Paper ruled accurately, either logarithmically or semi-logarithmically, can be obtained from Messrs. Pye, Cambridge, and probably from other makers, and is most useful for important work, but it is rather expensive for use in class, and these scales are intended to furnish a substitute.

Mathematical Apparatus, Instruments, etc.

We have received from *Messrs. George Philip & Sons* a box of wooden models dissected, which has been prepared by Mr. Thorold Gosset for the practical demonstration of various geometrical theorems. The six theorems which the models are primarily intended to demonstrate are: Euc. I. 36, 41, 38, 43, 47, and VI. 8, Cor. 1. They can also be used for superposition proofs, for proofs of I. 4, 8, 26, etc. The price of the box is four shillings, but the carpenter's shop or scissors and cardboard will in a quarter of an hour provide as many copies of the models as are required for class work, for it is of considerable importance that the models shall be handled by the pupil. We cannot help feeling that the class that has used these models will of its own free will and accord demand to construct its own models for Book II., and for other purposes. *Messrs. Lilley & Son*, Forest Works, Dalston, N.E., send a useful Radianometer of transparent celluloid In form it is like the ordinary protractor, but is graduated in radians and decimals of a radian, instead of in degrees, etc. Radians can be plotted accurately to two places of decimals, and to three places with considerable precision. It will be found especially useful for plotting harmonic curves, and curves representing the motion of slide valves. It is only one shilling. The same firm also provide a five-inch slide rule, in case, for seven shillings. They have a slide rule $10\frac{5}{8}$ inches long, with a clear space of $\frac{3}{8}$ of an inch at each end, enabling the cursor to be used firmly at both ends; a strip of celluloid at the back of the rule is introduced for the purpose of counteracting any tendency to warp. Their cheap boxwood slide rule in a case. at 4s. 6d., should have a sale in schools and technical institutes. It is fitted with a transparent celluloid cursor. They also send us a capital little snap case of instruments (No. 710), which should certainly be considered by those who are choosing cases of instruments for their classes. The case contains double-jointed Bow Compasses, fitted with fine sewing needles, with adjustable pen and pencil points and lengthening bar, divider, drawing pen, etc. The instruments are steel-jointed, and seem very good value for the money (7s. 6d.) We note that this firm proposes shortly to issue geometry models in celluloid. To these we look forward with great interest. *Mr. W. J. Brooks*, of 33 Fitzroy Street, W., sends us a novelty in the shape of a parabola curve in transparent celluloid, with the axis, focus, and latus rectum engraved on it. This will be found of use in teaching Geometrical Conics, and we may add that it is accurately constructed for use on squared paper, one inch unit. In teaching graphs its use is obvious. Mr. Brooks also sends pattern A of the Flexible Curves for draughtsmen. It is a flexible strip of celluloid, with tabs at intervals along its length, and is eight inches long. The curve is held down in the required position by the fingers on the tabs. The other patterns are self-clamping, reversible, and can be used for symmetrical drawings. These curves are made in several styles, according to the work for which they are required.

COMMISSIONS FOR LONDON UNIVERSITY CANDIDATES.

SPECIAL REQUIREMENTS FOR CANDIDATES FOR THE ROYAL ARTILLERY.

A University candidate for a commission in the Royal Artillery must produce satisfactory evidence that he has attained the standard of Mathematics required of a candidate for admission to the Royal Military Academy, which is as follows:—

(A.) ELEMENTARY MATHEMATICS.

(For the "Leaving" or "Qualifying" certificate.)

Arithmetic.—The ordinary rules, with applications more especially to the mensuration of plane figures and solids. The metric system and the use of decimals in approximate calculation, with contracted methods, will be specially insisted upon.

Neither the extraction of the cube root nor the use and theory of recurring decimals is required.

Neatness and accuracy of working are expected; and the methods of solution employed must be clearly indicated. There will be no objection to the intelligent use of algebraic formulæ and symbols.

Geometry.—The elements of geometrical drawing and practical geometry.

The substance of Euclid, Books I., II., III., and a working knowledge of as much of the properties of similar figures as is necessary for plan-making, map-drawing, and simple problems in mensuration.

Algebra.—To simple quadratic equations; the elementary use of graphs, particularly in connection with linear and quadratic functions.

The papers will be set to test knowledge of fundamental principles and readiness in application to simple practical problems. The solutions of equations must be worked out to a few significant figures; and candidates must be accustomed to test the accuracy of solutions by substitution. Skill in elaborate analysis, such as the simplification of complicated fractions, will not be looked for.

PRACTICAL WORK.

Measurement of length—verniers, calipers, micrometer, screw-gauge spherometer.

Measurement of angles—use of protractor, etc.

Measurement of areas—by dimensions, by squared paper, by weighing; area of cross-sections of a tube, calibration of a tube.

Measurement of volume—by dimensions, by graduated jar or burette.

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Principle of Archimedes—volume by weighing.

Specific gravity.

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(In the Competitive Examination for admission to the Royal Military Academy.)

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Proportion may be treated algebraically, and the complications of Euclid's definitions and nomenclature avoided.

Algebra.—As in elementary mathematics, together with indices and the simpler properties of surds; graphs of the simpler algebraic functions; theory of quadratic equations: use of graphs in solving equations, and in illustrating and solving practical questions: rate of variation of a function and gradient of a graph; graphic interpolation.

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THE
MATHEMATICAL GAZETTE.

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ON THE DEVELOPMENT OF MATHEMATICAL ANALYSIS AND ITS RELATIONS TO SOME OTHER SCIENCES.[*]

ONE of the objects of a Congress such as this is to exhibit the connecting links of the different departments of Science, taking that term in its widest acceptation. Hence the organisers of this meeting have taken care that the relations between the different sections shall be exhibited in broad relief. In undertaking a sketch of this kind, which must necessarily be somewhat indefinite in character, we must forget that all is in all; as far as algebra and analysis are concerned, the follower of Pythagoras would be disconcerted by the extent of his task, remembering the celebrated formula of the School: "Things are numbers." From this point of view my subject would be inexhaustible. But I shall make no such pretensions, and for the best of reasons. By passing in rapid review the development of our science throughout the ages, and particularly in the last century, I hope to be able to give an adequate indication of the part played by mathematical analysis in its relations to other sciences.

I.

I might naturally be expected to begin with the concept of the whole number, but this concept is not solely logical in its nature; it is also historical and psychological, and would involve us in too many discussions. Unfathomable depths have been revealed in the course of the study of the concept of number. For instance, no solution has yet been found of the problem under which of the two forms, the cardinal and the ordinal, does the idea of number first present itself, which of the two is anterior to the other, *i.e.* is the idea of number

[*] An address delivered at the Congress of Science and Art, St. Louis, U.S.A., September 22nd 1904, and translated for the *Gazette* by kind permission of the author.

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properly so called anterior to that of order, or *vice versâ*. In these questions geometers who are logicians seem to pay too little attention to psychology and the lessons to be derived from uncivilised races. Perhaps there is no general answer to the question, for the answer varies according to races and mental powers. In this connection I have sometimes thought of the distinction between auditives and visuals, the auditives being connected with the ordinal theory, and the visuals with the cardinal. But I must not linger on ground so full of pitfalls. I am afraid that our modern school of logicians has a difficulty in finding common ground with ethnographers and biologists; the latter, in questions of origin, are always dominated by the idea of evolution, and, to more than one among them, logic is but the summary of ancestral experience. Mathematicians have even been reproached for laying down in principle that there is a human mind in some measure external to things, and that it has a logic of its own. Whatever may be thought of this, the idea has been very useful, if not indispensable, to the progress of mathematical science; and certainly whether it evolved in the course of prehistoric ages or not, this logic of the human mind was firmly fixed in the oldest geometrical schools in Greece; the works of those schools appear to have been the first code of that logic, a fact indicated by the story of Plato writing on the door of his school: "Let none enter here if he be not a geometer."

A long time before the quaint word *Algebra* was derived from the Arabic, a word which expresses the operation by which equalities are reduced to a certain canonical form, the Greeks were unconsciously in the possession of algebra. We cannot even imagine relations closer than those which connect their algebra and their geometry, or rather, if need arose, it would be difficult to classify their geometrical algebra, in which they reasoned not on numbers, but on magnitudes. Among the Greeks, too, we find a geometrical arithmetic; and one of the most interesting phases of its development is the discussion which took place among the followers of Pythagoras on the difference between number and magnitude, *apropos* of irrationals. Although the Greeks cultivated the abstract study of numbers, which they called arithmetic, their speculative spirit found but little attraction in practical calculations, which they called logistic. In remote antiquity the Egyptians, Chaldeans, and later the Hindus and the Arabs, carried the science of computation to considerable lengths. To this they were forced by practical requirements. Logistic preceded arithmetic just as land surveying and geodesy opened the way for geometry. In a like manner trigonometry was developed under the influence of the increasing requirements of astronomy. The initial stages of the history of science show a close relation between pure and applied mathematics, and in the course of this address we shall constantly find this to be the case.

So far our domain has been that which is called in current language elementary arithmetic and algebra. But as soon as the incommensurability of certain magnitudes was recognised, the idea of infinity made its appearance, and from the days of Zeno's paradoxes on the impossibility of motion, geometrical progressions had to be dealt with. The processes of exhaustion which are found in Eudoxus and in Euclid really belong to the integral calculus, and Archimedes evaluates definite integrals. Mechanics also makes its appearance in his treatise on the quadrature of the parabola, for he first of all finds the area of the segment limited by an arc of the parabola and its chord, by means of the theorem of moments. This is the first example of those relations between mechanics and analysis which have since been continually developed. The infinitesimal method of the Greek geometers in treating volumes raised questions the interest of which has not even yet been exhausted. In plane geometry two equivalent polygons are equal by addition or subtraction, *i.e.* they may be decomposed into congruent triangles, or they may be regarded as the differences of polygons which may be so decomposed. It is not the same in the case of the geometry of space, and it has only recently been known that stereometry cannot, like planimetry, be independent of the processes of exhaustion or limit, which require the axiom of continuity or Archimedes' axiom. Without dwelling on this further, this glance into antiquity shows us how algebra, arithmetic, geometry, and elementary forms of the integral calculus and mechanics were intermingled to such an extent that it is impossible to separate their early history.

In the middle ages and in the renaissance the geometrical algebra of the ancients is separated from geometry. Little by little algebra, properly so called, became independent, with its symbolism and its notation more and more perfected. Thus is created this wonderfully clear language which economises thought and makes further progress possible. It is now also that distinct divisions become organised. Trigonometry, which in antiquity was but the handmaid of astronomy, begins to develop independently, and about the same time the logarithm appears, and essential elements are thus brought into evidence.

II.

In the 17th century the analytical geometry of Descartes—distinguished from what I just now called the geometrical algebra of the Greeks by the general and systematic ideas upon which it is founded—and the newly-born science of dynamics, proved to be the origin of very great progress in analysis. When Galileo, starting from the hypothesis that the velocity of heavy falling bodies is proportional to the time, deduces from it the law of the distance traversed, verifying it later by experiments, he is following the path already trodden by Archimedes, a path which was to be followed in time to come by Cavalieri, Fermat, and others, down to the time of Newton and Leibnitz The integral calculus of the Greek geometers is found again in the kinematics of the great Florentian physicist. As for the calculus of derivatives or differentials, it was laid down with precision in connection with the drawing of tangents. In reality, the origin of the idea of derivatives lies in the vague sense of the mobility of things and of the more or less extreme rapidity with which phenomena take place. This is well expressed in the terms *fluentes* and *fluxions* used by Newton, which we might suppose are borrowed from Heraclitus of old.

The standpoints of the founders of the science of motion, Galileo, Huygens, and Newton, had an enormous influence on the direction of mathematical analysis. In Galileo it was an intuition of genius to note that in natural phenomena the circumstances which determine motion produce acceleration. It was bound to lead him to lay down the principle that the rapidity with which the dynamical state of a system changes depends in a definite manner on its statical state alone. In a more general manner it was postulated that the infinitesimal changes, of whatever nature they may be, which take place in a system of bodies depend solely on the present state of that system. How far are the exceptions apparent or real? This is a question which was only raised at a later stage, and one which I pass over for the moment. From the principles enunciated is disengaged a point of great importance to the analyst. The phenomena are expressed by differential equations—equations which may be formed when observation and experiment have detected in each category of phenomena certain physical laws. One understands the unbounded hopes to which such results must give rise. As Bertrand says in the preface to his *Traité*, the initial successes were at first so great that it might be supposed that all the difficulties of science were already surmounted; and it was thought that geometers, no longer being distracted by the elaboration of pure mathematics, would be able to turn their thoughts exclusively to the study of natural laws. This was a gratuitous assumption that the problems of analysis presenting themselves would offer no very serious difficulties. But in spite of the disillusions which the future was to bring, the important point remained that these problems had assumed a precise form, and that the difficulties which were to be surmounted could be classified. Here, then, was an immense advance, one of the greatest that the human mind has ever made. In this manner we also learn why the theory of differential equations was to acquire considerable importance.

I have anticipated somewhat, in presenting things in such an analytical form. In all this progress geometry played its part. Huygens, for instance, always followed the ancients by preference, and his *Horologium Oscillatorium* is based on both infinitesimal geometry and mechanics. In the same way the methods followed in Newton's *Principia* are synthetic. But it is with Leibnitz in particular that science begins to move along those paths which were to lead it to what we call mathematical analysis. It is Leibnitz who in the last years of the 17th century is the first to employ the word *function*. By his power of systematisation, by the numerous problems which he and his disciples, James and John Bernouilli, treated, he showed definitively the power of the methods to which a long series of thinkers had in turn contributed ever since the far-off days of Eudoxus and Archimedes.

The 18th century showed the extreme fertility of the new methods. It was a curious age. It was an age of mathematical duels in which geometer launched defiance at geometer, in which the conflict was not always without bitterness, when the followers of a Leibnitz and a Newton entered the lists. From the purely analytical point of view the classification and the study of simple functions is particularly interesting; the idea of function on which analysis is based is thus gradually developed. Euler's celebrated researches at this time held an important position. However, the numerous problems which presented themselves to mathematicians scarcely left them time to scrutinise principles; the foundations of the doctrine are slowly elucidated, and the epigram attributed to d'Alembert: "Go on, and faith will come to you," is very characteristic of this age. Of all the problems raised at the end of the 17th century, or during the first half of the 18th, it will suffice to recall the problems of isoperimetry, which were to give birth to the calculus of variations. I prefer to dwell on the more intimate relations still between analysis and mechanics when, after the inductive period of the early days of dynamics, we reach the deductive period in which an endeavour was made to give a definite form to principles. Mathematical and formal development then played the essential rôle, and the language of analysis was indispensable to the widest extension of these principles. There are times in the history of sciences, and perhaps of societies, in which the mind is sustained and carried forward by the words and the symbols which it has created, and when generalisations present themselves with a minimum of effort. This was in particular the part played by analysis in the formal development of mechanics. Allow me here to interpose a remark. It is often said that there is nothing in an equation except what we put into it. It is easy to reply, in the first place, that the new form under which one finds things is often itself an important discovery. But we may go further. Analysis by the simple play of its symbols may suggest generalisations going far beyond the original limits. Thus it was with the principle of virtual velocities, the first idea of which sprang from the simplest mechanisms. The analytical form which translated it suggested extensions which led far from the point of departure. In one sense even, it is not fair to say that analysis has created nothing, for these more general conceptions are the work of analysis. We have another example in Lagrange's system of equations. Here the transformations of the calculus have given the type of differential equations to which we now tend to reduce the notion of mechanical explanation. In science there are few examples comparable with this of the importance of the form of an analytical relation, and of the power of generalisation of which it may be capable. It is very clear that in each case the generalisation suggested must be stated precisely by an appeal to observation and experiment. Finally, it is the calculus again which will seek out remote consequences to submit them to the same tests—but that is an order of ideas which I may not dwell upon here.

Under the impulse of the problems suggested by geometry, mechanics, and physics, we see developing, or springing into being, nearly all the great divisions of analysis. First we meet with equations with a single independent variable. Presently partial differential equations appear in the treatment of vibrating strings, of the mechanics of fluids, and of the infinitesimal geometry of surfaces. Here was quite a new analytical world. The very origin of the problems treated was of assistance and did not permit of much wandering from the right path in the early stages; and in the hands of Monge geometry rendered useful aid to these newly-born theories. But of all the applications of analysis none at that time was more brilliant than the problems of celestial mechanics, which were suggested by the knowledge of the laws of gravitation, and with which are connected the names of our most famous geometers. Theory never had a finer triumph; perhaps, even, it might be added that this triumph was too complete, for it was at this time especially that were conceived for the future of natural philosophy the premature hopes to which I have alluded During all this period, especially in the second half of the 18th century, what strikes us with admiration, and also is the cause of some little confusion, is the extreme importance of the applications realised, at a time when the foundations of pure theory appeared so uncertain. We see it when certain questions are raised, such as the degree of arbitrariness in the integral of vibrating strings, which gives rise to an interminable and somewhat inconclusive discussion. Lagrange realised these insufficiencies when he published his theory of analytical functions in which he endeavours to give precise foundations to analysis. It is impossible to admire too highly the wonderful presentiment he had of the part that the functions which we now, with him, call analytical functions were to play in the future. But we stand astonished, it must be confessed, when we read the proof that he thought he had given of the possibility of the development of a function in Taylor's series. Requirements in questions of pure analysis were but few and far between in those days. Trusting in intuition, men were content with certain probabilities, and agreed implicitly on certain hypotheses which it seemed unnecessary to formulate explicitly. In reality, they trusted in the solidity of ideas which had shown themselves fertile on so many occasions, and this is practically what d'Alembert expressed in his epigram. The necessity of rigour in mathematics has had its successive approximations, and in this respect our science does not possess the absolute character attributed to it by so many.

III.

We have now reached the early years of the 19th century. As I explained before, the great majority of analytical researches in the 18th century were occasioned by problems in geometry, and especially in mechanics or in physics, and so far we have found but little trace of the logical and aesthetic tendencies which are to give such a different appearance to so many mathematical treatises, especially in the last two-thirds of the 19th century. But I must not anticipate. After numerous examples of the influence of physics on the development of analysis, we shall meet with yet another, and that one of the most memorable, in Fourier's theory of heat.

Fourier begins by forming the partial differential equations which govern temperature. What are for such equations the conditions at the limits which enable us to obtain a solution? To Fourier's mind the conditions were suggested by the physical problem, and the methods he followed have served as models to the physicist-geometers of the first half of the last century. One of them consists in forming series with certain simple solutions. Fourier thus obtained his first types of development more general than the trigonometrical developments, as in the problem of the cooling of a sphere in which he applies his theory to the terrestrial globe, and seeks the law of variations of temperature in the soil, trying to obtain numerical applications.

In the presence of these grand results we can understand the enthusiasm of Fourier which glows in every line of his introductory remarks. Speaking of mathematical analysis, he says: "There can be no more universal or more simple language, no language more exempt from error and obscurity, *i.e.* more worthy to express the invariable relations of natural objects. Considered from this point of view it is as far extended as nature itself; it defines all sensible relations, it measures time, space, force and temperature. This difficult science is formed slowly, but it never loses a principle that it has once acquired. It ever grows and becomes strong, though surrounded by errors of the human mind." This praise is magnificent, but here we see signs of the tendency which makes of analysis but an auxiliary, however incomparable, to the natural sciences, a tendency of thought in conformity, as we have seen, with the development of science during the two preceding generations; but here we reach an epoch at which new tendencies make their appearance. Poisson, in a report on the *Fundamenta,* recalled the reproach levelled by Fourier at Abel and Jacobi, that they had not by preference devoted their attention to the motion of heat. Jacobi wrote to Legendre: "It is true that M. Fourier believed that the main aim of mathematics was public utility and the explanation of natural phenomena; but a philosopher of his ability ought to have known that the sole aim of science is the honour of the human intellect, and that on this ground a problem in numbers is as important as a problem on the system of the world." No doubt this was also the opinion of the great geometer of Göttingen, who called mathematics the queen of the sciences and arithmetic the queen of mathematics. It would be ridiculous to consider these two tendencies as directly opposed; the harmony of our science is in their synthesis. The moment was to arrive at which the necessity of inspecting the foundations of the edifice became evident, at which an inventory had to be made of the riches accumulated, adopting at the same time a more critical attitude. Mathematical thought was to acquire fresh impetus by a rigorous examination of mathematical principles. The problems are for a time exhausted, and it is not well that all investigation should proceed along one and the same line. Besides, differences and paradoxes which remained unexplained required further progress in pure theory. The path along which pure theory was to move was traced in its broad lines, and it could continue its journey independently, without necessarily losing contact with the problems suggested by geometry, mechanics, and physics. At the same time more interest was to be attached to the philosophical and artistic side of mathematics, trusting in a kind of pre-established harmony between our logical and aesthetic satisfactions and the requirements of future applications.

Let us rapidly recall a few points in the history of the revision of principles in which Gauss, Cauchy, and also Abel were workers at this early stage. Precise definitions of continuous functions and their most immediate properties, and simple rules for the convergence of series were formulated; and there soon was established under very general conditions the possibility of trigonometrical development, thus giving legal sanction to the boldness of Fourier. Certain geometrical intuitions relating to areas and arcs gave place to rigorous proofs. The geometers of the eighteenth century had necessarily tried to take into account the degree of generality of the solution of ordinary differential equations. Their resemblance to equations of finite differences readily led to the result; but the proofs obtained in this way will not bear close examination. Lagrange in his *Leçons sur le Calcul des Fonctions* had introduced more precision, and starting from Taylor's series, he saw that the equation of order m leaves indeterminate the function and its $m-1$ first derivatives for the initial value of the variable; but we are not surprised that Lagrange has not considered the question of convergence. In 20 or 30 years, demands for rigour in proofs had grown. We know that the two preceding methods of demonstration are susceptible of all the precision that is requisite. As for the first, there was no need of any new principle; and as for the second, the theory had to be developed along new lines. So far, functions and variables had been real. The consideration of complex variables now extended the field of analysis. The functions of a complex variable with a single derivative are necessarily developable in a Taylor's series. Thus we fall back on that mode of development, the interest of which had been recognised by the author of the *Théorie des Fonctions analytiques*, but the importance of which could not be fully realised as long as we limited ourselves to real variables. They also owe the great part they have not ceased to play to the facility with which they can be handled, and their convenience in calculations. The general theorems of the theory of analytical functions afforded a precise answer to questions which until then had remained undecided, such as the degree of generality of the integrals of differential equations. It became possible to complete the proof sketched by Lagrange for an ordinary differential equation; and precise theorems were established for a partial differential equation, or a system of such equations. The results obtained, important as they are, do not completely solve the different questions that may be raised; for in mathematical physics, and even in geometry, the conditions for the limits are susceptible of forms so varied that the problem known as Cauchy's problem often appears in a very restricted form. I shall come back later to this important point. EMILE PICARD.

(*To be continued.*)

I.—EXPANSIONS OF TRIGONOMETRICAL FUNCTIONS.

In the following articles the method of Averages, used in the Oct. 1903 issue of the *Gazette* to effect the expansion of the simple Algebraical functions, is employed for the Trigonometrical Expansions and for Taylor's Theorem, and the results are applied to the question of convergence. The method was adopted as the simplest I could devise for even the Binomial, subject to the condition that no infinite series should be used unless the limits of error after a finite number of terms were explicitly stated. I would suggest that this condition might well be laid down in a school course, as the uncertainty that comes over our Algebra as soon as we leave Geometrical Progressions 'to infinity' delays a clear grasp of Mathematics more than a fuller consideration of the simpler infinite series would. The method is, of course, essentially integral calculus, but I have found—and I hope my subjects have not been specially favourable—that the proofs can be readily followed by students to whom the ideas and notation of the Differential and Integral will not be familiar for a considerable time, and in whom perhaps they will never inspire a sense of absolute security.

1. *Expansion of the sine.*

$$\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2},$$

which

$$= 2\tan\frac{x-y}{2}\cos\frac{x-y}{2}\cos\frac{x+y}{2}.$$

Now, if the angles are in circular measure,

$$2\sin\frac{x-y}{2} < (x-y), \quad \text{and } 2\tan\frac{x-y}{2} > (x-y).$$

Moreover, if x and y are less than $\frac{\pi}{2}$,

$$\cos\frac{x+y}{2} < \cos y, \quad \text{and } \cos\frac{x-y}{2}\cos\frac{x+y}{2} = \tfrac{1}{2}(\cos x + \cos y) > \cos x;$$

$$\therefore\ \sin x - \sin y \overset{*}{<} (x-y)\begin{vmatrix}\cos y\\ \cos x\end{vmatrix}.$$

Now consider the interval between 0 and a broken up into N equal intervals, the intermediate angles being

$$\frac{a}{N}, \frac{2a}{N}, \dots \frac{(N-2)a}{N}, \frac{(N-1)a}{N},$$

which we may write shortly

$$a_1, a_2, \dots a_{N-2}, a_{N-1}.$$

---

* "Lies between."

Then, $\sin a \equiv (\sin a - \sin a_{N-1}) + (\sin a_{N-1} - \sin a_{N-2}) + \dots + (\sin a_1 - \sin 0)$,

that is,
$$\sin a = \sum_{i=1}^{i=N} (\sin a_i - \sin a_{i-1}).$$

Therefore, provided $a < \frac{\pi}{2}$ (so as to ensure that $\cos a_{i-1}$ is always greater than $\cos a_i$),
$$\sin a < \sum_1^N \frac{a}{N} \left| \begin{matrix} \cos a_{i-1} \\ \cos a_i \end{matrix} \right|,$$

that is,
$$< \frac{a}{N} \left| \begin{matrix} \cos 0 + \cos a_1 + \cos a_2 + \dots + \cos a_{N-2} + \cos a_{N-1} \\ \cos a_1 + \cos a_2 + \dots \qquad + \cos a_{N-1} + \cos a \end{matrix} \right|. \quad (1)$$

Now the difference between the upper and lower limits is
$$\frac{a}{N}(\cos 0 - \cos a) \text{ or } \frac{a}{N} \cdot 2 \sin^2 \frac{a}{2}, \text{ which } < \frac{a^3}{2N}.$$

But N is quite independent of all the other magnitudes concerned, and therefore we may consider it indefinitely increased, and may neglect this small error, writing for either of the limits in (1) $a \cdot \mathop{\mathrm{Av}}\limits_{\theta=0}^{\theta=1} (\cos \theta a)$, θ representing the fractions $\frac{1}{N}, \frac{2}{N}, \dots$ and 'Av' the average of the values when N is *indefinitely* increased.

Thus
$$\sin a = a \cdot \mathop{\mathrm{Av}}\limits_0^1 \cos \theta a. \quad \text{(s)}$$

An exactly similar piece of work leads to
$$\cos a - 1 = -a \cdot \mathop{\mathrm{Av}}\limits_0^1 \sin \theta a. \quad \text{(c)}$$

Hence, substituting for $\cos \theta a$ in (s) by means of (c),
$$\sin a = a \mathop{\mathrm{Av}}\limits_{\theta=0}^{\theta=1} \left[1 - \theta a \mathop{\mathrm{Av}}\limits_{\theta'=0}^{\theta'=1} \sin \theta\theta' a \right] = a - a^2 \cdot \mathop{\mathrm{Av}}\limits_0^1 \theta \cdot \mathop{\mathrm{Av}}\limits_0^1 \sin \theta\theta' a.$$

Using the same process on $\sin \theta\theta' a$, and introducing subscripts to θ, we get
$$\sin a = a - a^3 \cdot \mathop{\mathrm{Av}}\limits_0^1 \theta_1^2 \cdot \mathop{\mathrm{Av}}\limits_0^1 \theta_2$$
$$+ a^4 \cdot \mathop{\mathrm{Av}}\limits_0^1 \theta_1^3 \cdot \mathop{\mathrm{Av}}\limits_0^1 \theta_2^2 \cdot \mathop{\mathrm{Av}}\limits_0^1 \theta_3 \cdot \mathop{\mathrm{Av}}\limits_0^1 (\sin \theta_1\theta_2\theta_3\theta_4 a).$$

Now, by a simple process,* we find that $\mathop{\mathrm{Av}}\limits_0^1 \theta^n = \frac{1}{n+1}$;

$$\therefore \sin a = a - \frac{a^3}{\lfloor 3} + a^4 \cdot \text{(the same expression as before).}$$

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* *Gazette*, July, 1903, § v.

By repetition of the process we obtain, if n is odd,

$$\sin a = a - \frac{a^3}{\underline{|3}} + \frac{a^5}{\underline{|5}} - \dots \pm \frac{a^n}{\underline{|n}} \mp \text{ an error which is}$$

$$a^{n+1} . \operatorname{Av} \theta_1{}^n . \operatorname{Av} \theta_2{}^{n-1} \dots \operatorname{Av} \theta_n . \operatorname{Av} \sin (\theta_1 \theta_2 \dots \theta_n \theta_{n+1} a).$$

The value of this error can be obtained by the consideration that, as every θ is a fraction, the sine that occurs is less than $\sin a$.

$\therefore$ the error lies between $\dfrac{a^{n+1}}{\underline{|n+1}} \sin a$ and 0.

By the use of equation (s) above, it is equally easy to show that the error lies between $\dfrac{a^{n+2}}{\underline{|n+2}} \left| \begin{matrix} 1 \\ \cos a \end{matrix} \right|$, n still being odd.

In either case it diminishes indefinitely as n increases.

2. *The expansion of the cosine* follows from equation (c), or can be established *à priori* by similar work.

$$\cos a = 1 - \frac{a^2}{\underline{|2}} + \frac{a^4}{\underline{|4}} - \dots \pm \frac{a^n}{\underline{|n}} \ (n \text{ even}) \mp \text{ an error}$$

which lies between $\dfrac{a^{n+1}}{\underline{|n+1}} \left| \begin{matrix} \sin a \\ 0 \end{matrix} \right|$, or between $\dfrac{a^{n+2}}{\underline{|n+2}} \left| \begin{matrix} 1 \\ \cos a \end{matrix} \right|$.

[NOTE. For the sake of simplicity these expansions are here proved only for acute angles: it is shown in § II., that a slight modification of the proof makes it general.]

3. *Expansion of the inverse tangent.*

We have $\tan x - \tan y = \dfrac{\sin (x-y)}{\cos x . \cos y} = \tan (x-y) . \dfrac{\cos (x-y)}{\cos x \cos y}$.

Hence, $\tan x - \tan y < (x-y) \left| \begin{matrix} \sec^2 x \\ \sec^2 y \end{matrix} \right|$.

[We could get from this, as before,

$$\tan a = a . \overset{1}{\underset{0}{\operatorname{Av}}} (\sec^2 \theta a) = a + a^2 \overset{1}{\underset{0}{\operatorname{Av}}} (\tan^2 \theta a);$$

but the square under the average introduces difficulties.]

Since we may write the above in the form

$$(x-y) < (\tan x - \tan y) \left| \begin{matrix} \cos^2 y \\ \cos^2 x \end{matrix} \right|,$$

we have $\qquad \tan^{-1} a - \tan^{-1} b < (a-b) \left| \begin{matrix} (1+b^2)^{-1} \\ (1+a^2)^{-1} \end{matrix} \right|$

leading, by the usual process, to $\tan^{-1} x = x . \overset{1}{\underset{0}{\operatorname{Av}}} (1 + \theta^2 x^2)^{-1}$.

Now $\dfrac{1 \mp \theta^{2n}x^{2n}}{1+\theta^2x^2} = 1 - \theta^2x^2 + \ldots \pm \theta^{2n-2}x^{2n-2}$, the sign depending on n even or odd.

$\therefore\ \tan^{-1}x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} \ldots \mp \dfrac{x^{2n-1}}{2n-1} \pm$ an error $x^{2n+1} . \operatorname{Av} \dfrac{\theta^{2n}}{1+\theta^2x^2}$.

This error is less than*

$$x^{2n+1} . \operatorname{Av} \theta^{2n} . \operatorname{Av} (1+\theta^2x^2)^{-1},$$

which $$= \frac{x^{2n+1}}{2n+1} \cdot \frac{\tan^{-1}x}{x},$$

and is greater than $$\frac{x^{2n+1}}{2n+1} \cdot \frac{1}{1+x^2},$$

and therefore diminishes indefinitely if x is less than 1.

4. *Expansion of the inverse sine and cosine.*

Since $$\sin x - \sin y < (x-y) \left| \begin{matrix} \cos y \\ \cos x \end{matrix} \right|,$$

$$\sin^{-1}a - \sin^{-1}b < (a-b) \left| \begin{matrix} (1-a^2)^{-\frac{1}{2}} \\ (1-b^2)^{-\frac{1}{2}} \end{matrix} \right|.$$

Hence, as before, $\sin^{-1}x = x . \overset{1}{\underset{0}{\operatorname{Av}}} (1-\theta^2x^2)^{-\frac{1}{2}}$.

We here need the error after the n term of a Binomial Expansion.

In the case of $(1+u)^p$ (see §§ VI, VIII, *Gazette*, Oct. 1903) it lies between 0 and

$$(p-1)_n . u^n\{(1+u)^n - 1\},$$

$$(p-1)_n \text{ denoting } \frac{(p-1)(p-2)\ldots(p-n)}{\underline{|n}},$$

therefore the error in the case of $(1-\theta^2x^2)^{-\frac{1}{2}}$,

$$< (-\tfrac{3}{2})_n(-\theta^2x^2)^n \left| \begin{matrix} (1-\theta^2x^2)^{-\frac{1}{2}} - 1 \\ 0 \end{matrix} \right|;$$

$\therefore\ \sin^{-1}x = \displaystyle\sum_0^n \frac{1 . 3 \ldots \overline{2n-1}}{2 . 4 \ldots 2n} \cdot \frac{x^{2n+1}}{2n+1}$ + an error which lies between

$$\frac{3 . 5 \ldots \overline{2n+1}}{2 . 4 \ldots 2n} \cdot x^{2n+1} . \operatorname{Av} \theta^{2n}\{(1-\theta^2x^2)^{-\frac{1}{2}} - 1\} \text{ and } 0.$$

But $(1-\theta^2x^2)^{-\frac{1}{2}} < (1-x^2)^{-\frac{1}{2}}$;

$$\therefore \text{ the error} < \frac{1 . 3 . 5 \ldots 2n-1}{2 . 4 . 6 \ldots 2n} x^{2n+1}\{(1-x^2)^{-\frac{1}{2}} - 1\},$$

which diminishes indefinitely as n increases, if $x < 1$.

---

*The following inequality can be easily established: $\operatorname{Av} . ab . \gtrless \operatorname{Av} a . \operatorname{Av} b$ according as the a's and b's increase together or contrariwise.

5. *The hyperbolic functions.*

Exactly similar work serves for these: for

$$\sinh x - \sinh y = 2\sinh\frac{x-y}{2}\,.\cosh\frac{x+y}{2}$$

$$= 2\tanh\frac{x-y}{2}\,.\cosh\frac{x-y}{2}\cosh\frac{x+y}{2},$$

and therefore $$< (x-y)\left|\begin{matrix}\cosh x\\ \cosh y\end{matrix}\right|.$$

$$\cosh x - \cosh y = 2\sinh\frac{x-y}{2}\sinh\frac{x+y}{2} < (x-y)\left|\begin{matrix}\sinh x\\ \sinh y\end{matrix}\right|,$$

$$\tanh x - \tanh y = \frac{\sinh(x-y)}{\cosh x\cosh y} < (x-y)\left|\begin{matrix}\operatorname{sech}^2 y\\ \operatorname{sech}^2 x\end{matrix}\right|.$$

The method suggests the following answer to the difficult question 'what does "expansion" mean?'

The idea of continuity leads naturally to the expression of the increase of a function as a definite integral: expansion in a power series is the simplest form of integration by parts.

II.—EXPANSION OF FUNCTIONS IN GENERAL.

To prove that if f represents a function satisfying certain conditions for all values of the variable from a to $a+x$, then $f(a+x)$ can be expanded in powers of x.

Assume that, as in § I., $fu - fv < (u-v)\left|\begin{matrix}f'u\\ f'v\end{matrix}\right.$ where f' is another function which we call the *derived* function of f.

Now in the special cases investigated above $f'u$ was always greater or always less than $f'v$; in other words, f' was throughout an increasing or throughout a diminishing function. But this condition may not be fulfilled (*e.g.* $\cos x$ is diminishing from 0 to π, increasing from π to 2π, diminishing from 2π to 3π, etc.). In this general article slight modifications are introduced to provide for such cases.

If $x_1, x_2 \ldots$ are *any* intermediate values of the variable between x_0 and x_N, we obtain as before $fx_N - fx_0 < \Sigma(x_i - x_{i-1})\left|\begin{matrix}f'x_i\\ f'x_{i-1}\end{matrix}\right|$; but the upper limit sometimes has $f'x_i$ and sometimes $f'x_{i-1}$.

Now choose the simplest series of values for x_i. Let x_0 be a, and let x_i be $a+\frac{ix}{N}$, so that the intermediate values divide the whole range a to $a+x$ into N equal parts, and $x_i - x_{i-1} = \frac{x}{N}$.

Then $f(a+x)-fa<\frac{x}{N}\Sigma\left|\begin{matrix}f'x_i\\ f'x_{i-1}\end{matrix}\right|+\frac{x}{N}\Sigma'\left|\begin{matrix}f'x_{i-1}\\ f'x_i\end{matrix}\right|$, Σ including all values of i for which f' is increasing, and Σ' all values for which it is diminishing.

Now assume that the whole range consists of parts such that in the first f' is increasing, in the second diminishing, and so on alternately.

Then in any one of these parts the difference between the upper and lower limits of $\Sigma\frac{x}{N}\left|\begin{matrix}f'x_i\\ f'x_{i-1}\end{matrix}\right|$ is $\frac{x}{N}(f'x_m \backsim f'x_n)$ where x_m, x_n are the ends of the part.

If therefore f' is finite throughout and if the number of parts is finite, this difference between the sums vanishes when N is indefinitely increased.

Thus $f(a+x)-fa=\frac{x}{N}\Sigma f'x_i$ or, when N is infinite, $x.\overset{1}{\underset{0}{\mathrm{Av}}}\,f'(a+\theta x)$;

$$\therefore\ f(a+x)=fa+x\,\mathrm{Av}\,f'(a+\theta x).\ \dots\dots\dots\dots\dots(1)$$

If f' satisfies the same conditions as have been laid down for f,

$$f'(a+\theta x)=f'a+\theta x\,.\overset{\theta'=1}{\underset{\theta'=0}{\mathrm{Av}}}\,f''(a+\theta'.\theta x).$$

Hence, as in § I.,

$$f(a+x)=fa+xf'(a)+\frac{x^2}{\underline{|2}}f''(a)+\dots+\frac{x^{i-1}}{\underline{|i-1}}f^{i-1}(a)+\text{an error}$$

$$x^i.\overset{\theta_1=1}{\underset{\theta_1=0}{\mathrm{Av}}}\,\theta_1{}^{i-1}.\overset{1}{\underset{0}{\mathrm{Av}}}\,\theta_2{}^{i-2}.\dots.\overset{1}{\underset{0}{\mathrm{Av}}}\,\theta'_{i-1}.\overset{\theta_i=1}{\underset{\theta_i=0}{\mathrm{Av}}}\,f^i(a+\theta_1\theta_2\dots\theta_i x).$$

Now, $\theta_1\theta_2\dots\theta_i$ being a fraction, $a+\theta_1\theta_2\dots\theta_i x<\left|\begin{matrix}a+x\\ a\end{matrix}\right|$;

$\therefore$ the f^i lies between the greatest and least values of f^i in the range;

$$\therefore\ \text{the error}<\frac{x^i}{\underline{|i}}\left|\begin{matrix}A\\ B\end{matrix}\right|.$$

If this error vanishes when i is increased indefinitely, $f(a+x)$ can be expanded in powers of x.

Conditions. It will be seen that the conditions to which f is subject are reducible to the following:

(1) $fu-fv$ must lie between $(u-v)f'u$ and $(u-v)f'v$, at least when the difference $(u-v)$ is infinitely small, and the same condition must hold for the derived functions f', f'', ...

(2) $\frac{x}{N}(f'x_m \backsim f'x_n)$ must vanish for infinite N when x_m, x_n are the ends of increasing or diminishing periods of f'. (This will be satisfied if f' is everywhere finite.)

(3) The number of these periods must be finite; and conditions (2) and (3) must hold for all the derived functions.

A function satisfying these conditions is called 'continuous.'

If f' is always increasing or always diminishing f might be called 'simply continuous.'

Corollary 1. If $f'x$ is given over a range $x=a$ to $x=b$, $fb-fa$ is determined by (1) without ambiguity, and therefore fx is determined if its value at any one point of the range is fixed arbitrarily.

Corollary 2. The geometrical interpretations of the conditions are simple and interesting; but the ideas have intentionally been kept analytical. It will be noticed that 'points of inflexion' do not of themselves break 'continuity' or interfere with the expansibility of a function.

Corollary 3. To prove that, with certain conditions,

$$f\phi(a+x)-f\phi a=x\,.\underset{0}{\overset{1}{\mathrm{Av}}}\,f'\phi(a+\theta x)\,.\,\phi'(a+\theta x).$$

This means that, in the general inequality at the beginning of the article, x_i is taken to be any function ϕ of $\left(a+\frac{ix}{N}\right)$.

We have $$x_i-x_{i-1}<\frac{x}{N}\begin{vmatrix}\phi'_i\\ \phi'_{i-1}\end{vmatrix}.$$

If $\phi'_i\gtrless\phi'_{i-1}$ when $f'\phi_i\gtrless f\phi'_{i-1}$, no difficulty arises, but, consider that for some successive values of i this is not true.

In such a period, if $\phi'_i<\phi'_{i-1}$ while $f'\phi_i>f'\phi_{i-1}$,

$$f\phi_i-f\phi_{i-1}<\frac{x}{N}\begin{vmatrix}\phi'_{i-1}\,.\,f'\phi_i\\ \phi'_i\,.\,f'\phi_{i-1}\end{vmatrix},$$

$$i.e.\ <\frac{x}{N}\begin{vmatrix}\phi'_i\,.\,f'\phi_i+(\phi'_{i-1}-\phi'_i)\,.\,f'\phi_i\\ \phi'_{i-1}\,.\,f'\phi_{i-1}-(\phi'_{i-1}-\phi'_i)\,.\,f'\phi_{i-1}\end{vmatrix};$$

$\therefore$ if A is the greatest value of $f'\phi_i$ in the period,

$$f\phi_m-f\phi_n<\frac{x}{N}\Sigma\begin{vmatrix}\phi'_i\,.\,f'\phi_i+\\ \phi'_{i-1}\,.\,f'\phi_{i-1}-\end{vmatrix}(\phi'_{i-1}-\phi'_i)A\Bigg|,$$

where m, n are the values of i at the ends of the period;

and $$\frac{x}{N}\Sigma(\phi'_{i-1}-\phi'_i)A\equiv\frac{Ax}{N}(\phi'_m-\phi'_n)$$

which vanishes if ϕ' is always finite.

Hence $$f\phi(a+x)-f\phi a=x\,.\underset{0}{\overset{1}{\mathrm{Av}}}\,f'\phi(a+\theta x)\,.\,\phi'(a+\theta x)$$

if $f'\phi$ and ϕ' are finite through the range. W. N. ROSEVEARE.

(*To be continued.*)

CONSTRUCTIONS WITH STRAIGHT-EDGE AND DIVIDERS.

Problem 1. From a given point A of a straight line AB to draw a line making with AB an angle equal to a given angle CAD which has its vertex at A.

From AB and AD mark off equal lengths AE, AF; and let EF cut AC in G.

From FE produced mark off $EH=GF$. Then AH is the required line, the angle BAH being equal to the angle CAD in magnitude and sign.

The measure of the complexity of this construction is 13.

Problem 2. To bisect a straight line AB.

Through A draw any straight line, and mark off on it equal distances AC, CD, DE.

Produce EB to F making $BF=EB$.

Let CF cut AB in G. G is the mid point of AB.

Complexity $=15$.

A modification of this construction, having the same complexity, is obtained if F is taken in BE produced instead of in EB produced, and F is joined to D instead of to C.

Problem 3. To draw a parallel to a given line AB through a given point C.

Produce CA to D making $AD=CA$. Produce DB to E making $BE=DB$. CE is the required line.

Complexity $=15$.

Problem 4. At a given point A to erect a perpendicular to a given line AB.

Draw any line CBD cutting AB in B and make BC and BD each $=BA$. Produce CA to E making $AE=AB$.

Let EB and AD intersect in F. Mark off from EB the length $EG=FB$. AG is the required perpendicular.

Complexity $=24$.

Problem 5. From an external point C to draw a perpendicular to a given straight line AB.

Draw CD to meet AB in D. On either side of D mark off from the given line lengths DE, DF each $=DC$. From CE cut off $CG=CF$. Let GF cut CD in H. From FG mark off $FK=HG$. Let CK meet AB in L. CL is the required perpendicular.

Complexity $=22$.

Problem 6. To trisect a given straight line AB.

Through A draw ACD making $AC=CD$.

Produce DB to E making $BE=BD$.

Let CE intersect AB in F. From FA mark off $FG=FB$.

Then F and G are the points of trisection.

Complexity $=17$.

Problem 7. Given an angle BAC to make an equal angle at A on the opposite side of AB.

Through A draw any line AD and mark off $AE=AD$ on AB. Let ED meet AC in F. Produce DE to G making $EG=FD$. Join AG and from it cut off $AH=AE$. Let EH meet AD in K. Produce EH to L making $HL=KE$. Then AL makes with AB an angle equal to CAB.

Complexity = 28.

Problem 8. From a given point A in a given line AB to draw a line making with AB an angle equal to a given angle CDE.

Produce AD to N making $DN=AD$, and produce DA to F making $AF=DA$. Draw any line FGH through F cutting AB in G and make $GH=FG$. Mark off from DE, $DK=DH$. Let KH meet DC in L. From HK mark off $HM=LK$. Produce NM to P making $MP=NM$. Then AP is the required line.

Complexity = 32.

Note. The above solutions aim at being as simple as possible from the point of view of *construction*, taking the same *postulates* as those employed in Dr. Macaulay's article in last October's *Gazette*. They are therefore quite different from those which would be chosen if one wished to make the *proofs* as simple as possible. The complexity of each construction has been estimated according to the principle of Lemoine's *Géométrographie*, counting *one* every time the straight edge is adjusted to one point, or a point of the dividers adjusted to a point or a line, and every time a line is drawn.

I would like to suggest in this connection that the distinction between the two aims in elementary geometry, on the one hand the demonstration of propositions, and on the other hand the discovery of constructions for given problems, might with advantage be more emphasized in our "New Geometries"; and I would further suggest that the latter is a better exercise for the tyro in geometry. Lemoine's numerical test of simplicity makes a convenient standard for roughly comparing the merits of two constructions.

This article has been suggested by that of Dr. Macaulay. The problems chosen are among the most fundamental ones that can be based on the chosen postulates. Euclid's propositions 2 and 3 of Bk. I. show that every problem soluble by the postulates of this article, is also soluble by Euclid's postulates; but the converse is, I think, not true.

Perhaps some reader of the *Gazette* may be able to send a simplest possible construction for the problem: *To draw by the aid of straight-edge and dividers alone, a line bisecting at right angles the line joining two given points*, in other words *To draw the radical axis of two points.* I have hitherto found no construction whose complexity is less than 38; while with *straight edge and compasses* a construction of complexity 7 exists, and with *Euclid's postulates* one of complexity 9.

[An old paper of mine on this subject, referred to by Dr. G. B. Halsted in his note in the March number of the *Gazette*, appeared not in the *Edin. Math. Soc.* Proceedings, but in those of the *Phil. Soc. of Glasgow.*] R. F. MUIRHEAD.

P.S.—Since writing the above, I have had an opportunity of glancing through Hilbert's *Grundlagen der Geometrie* (ed. 1899) for the first time, and I find that in that most interesting work the author gives some constructions based on the postulates used in the foregoing article. One of his constructions is virtually the same as the above solution of Problem 3, though by fixing points in the given line the complexity is slightly increased.

Professor Hilbert also gives a discussion of the question as to what kinds of problems of construction can be solved with these postulates, and arrives at a criterion, from which it would appear that Euclid I. 22 cannot be solved by straight-edge and dividers alone. I had previously become convinced of the impossibility just mentioned but without being able to give an *a priori* reason for it. The same impossibility would exist in the case of the simpler problem:—to draw a line of given length from a given point to a given straight line.

MATHEMATICAL NOTES.

170. [V. a.] *Negative quantities* (a very elementary note).

Certain difficulties, which history shews to be real and serious, in obtaining a correct idea of negative quantities are sometimes inadequately dealt with in books on elementary algebra.

The resulting confusion of thought is apt to set up mischief, as the doctors say, at a much later stage; as the following quotation from a recent school book on Physics sufficiently indicates: "If u is less than $\frac{1}{2}r$, then v is negative and equal to $-v$."

In arithmetic we perform certain operations on numbers. These numbers are, to begin with, positive integers. The operations of Addition and Multiplication always produce positive integers if we commence with positive integers. The operation of dividing one positive integer by another does not always furnish a positive integer. Instead, however, of saying that, for example, it is impossible to divide 7 by 3 we introduce the notion of a Fraction.

The operation of subtracting one positive integer from another does not always furnish a positive integer. Are we to say that $3-7$ or -7 by itself calls for the performance of an impossibility?

In generalised Arithmetic or Arithmetical Algebra our symbols a, b, c, etc., still denote essentially positive quantities, and the same difficulty of interpretation occurs.

We next perceive that an extension of meaning of the signs + and − will enable us to attach a meaning to -7. The signs + and − may be taken to indicate opposite qualities in the objects represented. Thus if we are reckoning up a person's debts we may denote a sum of £5 due from him by $+5$; and then a sum of ten pounds due *to* him may be denoted by -10.

We can now give a meaning to the previously unintelligible expression $3-7$, and we perceive that whether a particular quality may more conveniently be denoted by the $+$ or by the $-$ sign depends on the point of view from which the problem is considered, and that the choice is a matter of convention.

It is evidently possible to employ a symbol to denote not only a quantity but also the sign of that quantity. Thus we might denote by the symbol x a quantity whose value proved to be -5. In this case x is a negative quantity.

The advantage of employing symbols which contain their own sign is brought out by contrasting the statements.

ARITHMETIC.

A owes me £a. B owes me £b. I owe C £c.

The symbols a, b, c, all denote positive quantities.

On these transactions I am in debt £$(c-a-b)$ or in credit by £$(a+b-c)$ according as c or $a+b$ is the greater.

ALGEBRA.

Let the balances due to me from A, B, C be a, β, γ respectively; β, for instance, being negative and equal to $-b$ if in reality I owe £b to B. Then on these transactions I am worth £$(a+\beta+\gamma)$, and this expression adequately represents the following arithmetical results:

I am in credit on the transaction	I am in debt on the transaction
$a+b+c$	$a+b+c$
$a+b-c$	$a+b-c$
$a-b+c$	$a-b+c$
$-a+b+c$	$-a+b+c$
$a-b-c$	$a-b-c$
$-a-b+c$	$-a-b+c$
$-a+b-c$	$-a+b-c$

as the case may be.

The unsatisfactory character of the definition "a quantity with the $-$ sign prefixed is called a negative quantity" (Baker and Bourne, p. 7) is apparent, and may be illustrated by the problem, "In 1905 A is a years old, B is b years old. Find the year in which A might truly say to B, 'I am twice as many years old as you.'"

If x denotes the interval in years between the present year and the year in question, x is an algebraical symbol. Let x be positive if the year in question is in the future.

a and b denote essentially positive quantities. We obtain the equation

$$x=a-2b,$$

and we may take the numerical values

$$a=37,\ b=6;\quad a=37,\ b=35.$$

The weak point in the definition quoted above may be further exposed by applying it to $-\log \cdot 5$ or to $\cos 120^\circ$.

There is an interesting article in the *English Cyclopedia* by De Morgan on Negative Quantities, which anyone who finds a difficulty in clearing up his pupils' ideas on this point may perhaps find it worth while to consult.

C. S. JACKSON.

171. [L^2. 8. b.] *On the normals from a point to a conicoid.*

Consider the rectangular hyperbolas in the principal planes of the conicoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which pass respectively through the feet of the four normals from $(0, \eta, \zeta)$ to the section by $x=0$; the feet of the four normals from $(\xi, 0, \zeta)$ to the section by $y=0$; and the feet of the four normals from $(\xi, \eta, 0)$ to the section by $z=0$. Then it may be proved that the feet of the six normals to the conicoid itself from ξ, η, ζ lie on the common curve of intersection of the three right cylinders whose cross sections are these three rectangular hyperbolas.

For the six feet are the intersections of the conicoid with

$$\frac{\xi-x}{x/a^2}=\frac{\eta-y}{y/b^2}=\frac{\zeta-z}{z/c^2},$$

i.e. with
$$\frac{\eta-y}{y/b^2}=\frac{\zeta-z}{z/c^2}; \quad \frac{\zeta-z}{z/c^2}=\frac{\xi-x}{x/a^2}; \quad \frac{\xi-x}{x/a^2}=\frac{\eta-y}{y/b^2}.$$

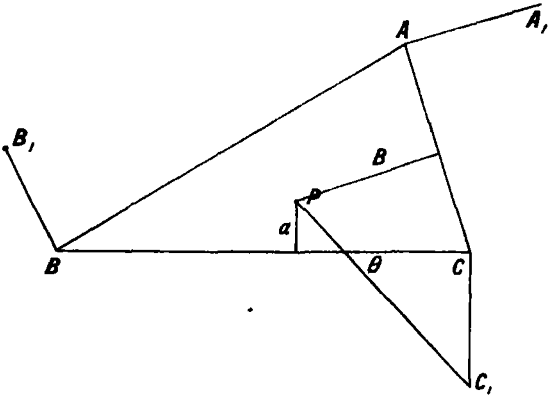
But these are the three cylinders mentioned above, and are seen to have a common curve of intersection. Thus the proposed theorem follows.

The centre of the conicoid and the point ξ, η, ζ itself are also on this curve.
H. L. TRACHTENBERG.

172. [K. 2. d.] *Three conics connected with a triangle.*

At C, A, B draw CC_1, AA_1, BB_1 respectively perpendicular to BC, CA, AB, such that $CC_1=AA_1=BB_1=v$.

Then as v varies, the locus of P, the centre of the circle $A_1B_1C_1$, is a cubic. For if P be α, β, γ, and PC_1 make θ with BC, and R be the radius PC_1,



$$\alpha+v=R\sin\theta \quad \text{and} \quad R\cos\theta=\frac{\beta+\alpha\cos C}{\sin C};$$

$$\therefore \quad R^2=(\alpha+v)^2+\frac{(\beta+\alpha\cos C)^2}{\sin^2 C}.$$

$\therefore$ since two similar equations are true, we have, on eliminating R^2,

$$\alpha^2+2\alpha v+\frac{(\beta+\alpha\cos C)^2}{\sin^2 C}=\beta^2+2\beta v+\frac{(\gamma+\beta\cos A)^2}{\sin^2 A},$$

$$\alpha^2+2\alpha v+\frac{(\beta+\alpha\cos C)^2}{\sin^2 C}=\gamma^2+2\gamma v+\frac{(\alpha+\gamma\cos B)^2}{\sin^2 B}.$$

∴ eliminating v, we have the required locus,

$$\Sigma(\beta-\gamma)\left(\frac{\alpha^2+\beta^2+2\alpha\beta\cos C}{\sin^2 C}\right)=0. \qquad\qquad (1)$$

Again, the corresponding locus for perpendiculars drawn to CB, BA, AC at B, A, C respectively, is

$$\Sigma(\gamma-\beta)\left(\frac{\alpha^2+\gamma^2+2\alpha\gamma\cos B}{\sin^2 B}\right)=0. \qquad\qquad (2)$$

These cubics intersect on the cubic

$$\Sigma(\beta-\gamma)\left(\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{\sin^2 A}\right)=0, \qquad\qquad (3)$$

as is seen by subtracting one left-hand side from the other.

These cubics each consist of the line at infinity and a conic, the three conics being

$$\Sigma\alpha\left\{\frac{\alpha+\gamma\cos B}{\sin B}-\frac{\beta+\gamma\cos A}{\sin A}\right\}=0;\quad \Sigma\alpha\left\{\frac{\alpha+\beta\cos C}{\sin C}-\frac{\gamma+\beta\cos A}{\sin A}\right\}=0;$$

$$\Sigma\alpha\left\{\frac{\beta+\alpha\cos C}{\sin C}-\frac{\gamma+\alpha\cos B}{\sin B}\right\}=0.$$

It will be noticed that the incentre and circumcentre lie on all three. Again, the bisector AI of the angle A, $\beta-\gamma$, meets the cubic (1) again in points on

$$\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{\sin^2 A}=\frac{\gamma^2+\alpha^2+2\gamma\alpha\cos B}{\sin^2 B},$$

i.e. on the straight line through the circumcentre, perpendicular to the side AB, and on the line at infinity.

Hence with similar results we have the fact that if AI meet the 3 lines through O perpendicular to BC, CA, AB respectively in

$$U_1,\ V_1,\ W_1,$$

and if the bisectors BI, CI meet them in

$$W_2,\ U_2,\ V_2$$

and V_3, W_3, U_3

then the points W_1, W_2, W_3 lie on the first conic,

the points V_1, V_2, V_3 „ the second conic,

and the points U_1, U_2, U_3 „ the third conic.

It will be noticed that P never goes to infinity in cases (1) and (2) in the conic parts of the loci. This can be seen geometrically (v being called negative when CC_1, BB_1, AA_1 are drawn in the opposite directions to those shewn in the figure). Thus the conics corresponding to (1) and (2) are closed.

It follows from the geometrical properties by which the cubics (1) and (2) were defined that there are 4 points connected with a triangle such that each is the centre of a pair of concentric circles, one of which cuts off equal portions CC_1, BB_1, AA_1 from the perpendiculars to CB, BA, AC at C, B, A respectively, and the other of which cuts off equal portions BB_1', CC_1', AA_1' from the perpendiculars to BC, CA, AB at B, C, A respectively; for the 4 points of intersection of the conics corresponding to (1) and (2) have this property.

H. L. TRACHTENBERG.

REVIEW.

Elements of Mechanics. Forty Lessons for Beginners in Engineering. By MANSFIELD MERRIMAN, Professor of Civil Engineering in Lehigh University. (New York, Wiley & Sons. London, Chapman & Hall. 1905. pp. 172. $1.)

Professor Merriman is the author of a number of text-books on several branches of applied mathematics, including a well-known treatise on the Method of Least Squares.

In the present volume his aim is to employ "the best methods of applied mechanics in the development of the fundamental principles and methods of rational mechanics." "To this end constant appeals are made to experience, by which alone the laws of mechanics can be established, ... and a system of units is employed with which every boy is acquainted." In the first 111 pages the leading principles of Statics are dealt with. The remainder of the book is devoted to Dynamics. Four hundred examples give ample occupation to the student.

The work is written in clear and concise terms, and the author appears thoroughly justified in expressing the hope that a student who has worked through the 400 problems will have laid a broad and strong foundation for his future progress. One point of detail offers an opening for discussion. Professor Merriman makes much use of the terms Inertia and Force of Inertia, following closely Newton's explanation of "Vis Inertiae." A teacher of elementary work has always to bear in mind that not everything which is lawful is expedient, and it may be permissible to remark that the expediency of using the terms Inertia, Force of Inertia, *in an elementary text-book*, is not established by the mere fact that Newton used them. Is not the statement "matter has Inertia" merely a poetical, or proverbial, or pithy mode of stating that Matter is distinguished from No-matter by the facts that Matter conforms to the "Laws of Motion"? Professor Merriman's own use of the terms Inertia and Force of Inertia is perfectly precise and unimpeachable; but does not the use made of so comfortable a word by some writers encourage a student to indulge in pseudo explanations such as that a body is hard to stop because it has so much inertia—a proposition which on investigation will be often found to belong to a transcendental branch of logic, known as feminine?

"Inertia" as an *explanation* of phenomena belongs to the class of philosophical terms which includes "vital force" as an *explanation* of the growth of living things. C. S. JACKSON.

ERRATA.

p. 181, line 10, *for* "portion" *read* "position."
p. 181, line 28, *for* "assumption" *read* "assumptions."
p. 181, line 2 up, *for* "diameter $x\pi$" *read* "diameter $\times \pi$."
p. 182, line 9, *for* "wave" *read* "move."
p. 182, line 23, *for* "vigorous" *read* "rigorous."

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Some Proofs of Newton's Theorem on Sums of Powers of Roots. A proof of Waring's Expression for Σa^r *in terms of the Coefficients of an Equation.* By R. F. MUIRHEAD. (Proc. Edin. Math. Soc. Vol. xxiii. 1904-5.) pp. 9.

Practical Uses of the Product $\mathbf{ab}^2$ *in Vector Analysis.* By J. V. COLLINS. pp. ii. From *Am. Math. Monthly*, vol. xii, No. 2. 1905.

Correlation of Vector Analysis Notations. By J. V. COLLINS. pp. 4. From *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Ap. 4, 1905.

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THE
MATHEMATICAL GAZETTE.

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WITH THE CO-OPERATION OF
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ON THE DEVELOPMENT OF MATHEMATICAL ANALYSIS AND ITS RELATIONS TO SOME OTHER SCIENCES.

IV.

Without restricting ourselves to the historical order, let us resume the development of mathematical physics in the last century, in so far as analysis is concerned. The problems of thermal equilibrium lead to the equation already known to Laplace in the study of attraction. There are few equations which have been the object of so many researches as this celebrated one. The conditions for the limits may assume various forms. The simplest case is that of the thermal equilibrium of a body, the elements of the surface of which are maintained at given temperatures. From the physical point of view, it may be regarded as evident that the temperature, assumed continuous in the interior since there is no source of heat, is determined when it is given at the surface. The more general case is that in which, the condition remaining permanent, there would be a radiation outward with an intensity varying at the surface according to a given law; in particular the temperature may be given over one portion, while there is radiation over the remainder. These questions, which are not yet solved in their widest generality, have largely contributed to the direction taken by the theory of partial differential equations. They have called attention to types of determination of the integrals which would never have presented themselves if we had been restricted to a purely abstract point of view. Laplace's equation has been already met with in hydrodynamics, and in the study of attraction varying inversely as the square of the distance. The latter theory brought to light elements of the most essential nature, such as the potential of single and double layers. Here we meet with analytical combinations of the highest importance, which have since been notably generalised. Green's formula is a case in point. The fundamental problems of electrostatics are of the same order of ideas, and certainly the celebrated theorem on electrical phenomena in the interior of a hollow conductor, which Faraday rediscovered at a later stage by experimental means, knowing nothing whatever of Green's memoir, was a notable triumph for theory. This magnificent aggregate has remained the type of the classical theories of mathematical physics, which seem to us to have almost attained perfection, and which have exercised, and still exercise, so happy an influence on the progress of pure analysis by suggesting to it the most beautiful problems. The theory of functions again will afford us a notable comparison. The analytical transformations brought into play are not distinct from those we have met with in the steady movement of heat. Certain fundamental problems in the theory of functions of a complex variable have lost their abstract enunciation and assumed a physical form, as in the case of the distribution of temperature on a closed surface of any connectivity whatever and without radiation, in thermal equilibrium, with two sources of heat which necessarily correspond to equal and opposite flows. Interpreting this, we find a question on Abelian integrals of the third species in the theory of algebraical curves.

The preceding examples, in which we have considered only the equations of heat and attraction, show that the influence of physical theories is exercised, not only on the general nature of the problems which have to be solved, but even in the details of analytical transformations. As an instance in point in recent memoirs on partial differential equations, we find under the name of Green's formula, a formula suggested by the original formula of the English physicist. The theory of electrodynamics and that of magnetism, in the hands of Ampère and Gauss, have also led to important progress. The study of curvilinear integrals and that of surface integrals owe their development to this theory, and formulas such as that of Stokes, which might also be called Ampère's formula, appeared for the first time in physical memoirs. The equations of the propagation of electricity, with which are connected the names of Ohm and Kirchoff, while presenting a marked analogy with those of heat, often offer slightly different limiting conditions; and we know all that cable telegraphy owes to the detailed discussion of the integrals of one of the Fourier equations applied to electricity. The equations long formulated in hydrodynamics, the equations in the theory of elasticity, and those of Maxwell and Hertz in electro-magnetism, afforded problems analogous to those mentioned above, but under still more varied conditions. There we meet with many as yet unsurmounted difficulties, but what magnificent results do we owe to the study of particular cases, the number of which we should like to see increased! There should also be noted, as concerning analysis and physics alike, the profound differences presented by propagation by methods varying according to the phenomena that are studied. In equations such as that of sound, we have propagation by waves. In the equations of heat every variation is felt instantaneously at any distance, but very slightly at a very great distance, and here we cannot speak of velocity of propagation. In other cases, of which Kirchoff's equation relative to the propagation of electricity with induction and capacity affords the simplest type, there is a wave front with a given speed, but with a residue behind which does not vanish. These different circumstances reveal very different properties of integrals; research into them has only taken place in a few particular cases, and it raises problems in which we have to deal with the profoundest ideas in modern analysis.

V.

I shall now enter into certain analytical details which are especially interesting in mathematical physics. The question of the generality of the solution of a partial differential equation presents what appear to be paradoxes. For one and the same equation the number of arbitrary functions figuring in the general integral is not always the same, but depends on the form of the integral in question. Thus Fourier, discussing the equation of heat in an unlimited medium, regards it as evident that a solution will be determined if we are given its value for $t=0$, *i.e.*, if we are given *one* arbitrary function of the three co-ordinates, x, y, z. From Cauchy's point of view we may consider, on the contrary, that there are in the general solution two arbitrary functions of three variables. In reality, the question in the form in which it has long been set has no precise signification. In the first place, when it is only a question of analytical functions, any finite number of functions whatever, with any number whatever of independent variables, does not present from the arithmetical point of view more generality than a single function of a single variable, since in both cases the aggregate of the coefficients of the developments forms an enumerable sequence. But there is something more. In reality, in addition to the conditions which are represented by given functions, an integral is subject to conditions of continuity, or must become infinite in a given manner for certain elements. We may thus be led to regard the condition of continuity in a given space as equivalent to an arbitrary function. And so we see how the question of the enumeration of the arbitrary functions is improperly formulated. It is sometimes a delicate matter to prove that conditions determine a solution uniquely, when we are unwilling to rest content with probabilities; in these cases we must state in exact terms the manner in which the function and certain of its derivatives behave. Thus, in Fourier's problem referring to an unlimited medium, certain hypotheses should be made as to the function and its derivatives at infinity, if we wish to establish that the solution is unique.

Formulas analogous to those of Green are very useful, but the proofs deduced from them are not always entirely rigorous, implicitly assuming conditions fulfilled for the limits which *à priori*, at least, are not necessary. This is one among several other instances of the evolution of exigency in the rigour of proofs. Let me also remark that the new investigation which is now necessary has often led to our being the better able to understand the nature of integrals; this is because true rigour is fertile, thereby being distinguished from rigour of another type, which is purely formal and distasteful, and which casts a shadow over the problems it touches.

Difficulties in the proof of the uniqueness of a solution may be very different, according as it is a question of equations of which all the integrals are or are not analytical. This is an important point, and one that shows that, even although one would like to avoid them, we must sometimes take into account functions that are not analytical. For instance, it cannot be affirmed that Cauchy's problem determines uniquely a solution, when the data of the problem are general—that is to say, when not characteristic. This is surely the case if we consider only analytical integrals, but in the case of non-analytical integrals there may be contacts of an infinite order. And here theory does not outstrip applications; the contrary is the case, as the following example will show. Does Lagrange's celebrated theorem on the velocity-potential in a perfect fluid hold good in a viscous fluid? Examples have been given[1] in which the co-ordinates of the different points of a viscous fluid, starting from rest, cannot be expressed in analytical functions of the time which has elapsed since the initial moment of the motion, and in which the rotations, as well as all their derivatives with respect to the time at that moment are null, and yet are not identically null; Lagrange's theorem therefore does not hold. These considerations sufficiently show the interest that may attach to our ascertaining whether all the integrals of a system of partial differential equations continuous, as well as all their derivatives, up to a given order in a certain field of real variables are analytical functions; it is understood that we assume there are only analytical elements in the equations. For linear equations we have precise theorems, all the integrals being analytical, if the characteristics are imaginary, and very general propositions have also been obtained in other cases. And let me remark again that the conditions for the limits which one has to assume are widely divergent, according as the integrals of the equation with which we are concerned are or are not analytical. A question of the first type is given by the generalised problem of Dirichlet. Conditions of continuity play in it an essential part, and in general the solution cannot be extended on both sides of the continuum with reference to which the data are defined. This is no longer so in the second case, in which the relation of this continuum with respect to the characteristics plays the principal part, and in which the field of existence of the solution is presented under quite different conditions. All these ideas, which are difficult to express accurately in ordinary language, are fundamental in mathematical physics, and are not of less interest in infinitesimal geometry. It will suffice to remember that all surfaces of constant positive curvature are analytical, while non-analytical surfaces of constant negative curvature exist.

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[1] In a Memoir by M. Boussinesq (*Comptes rendus*, Mar. 29 and Ap. 26, 1880).

From the earliest times a vague belief in a certain economy in natural phenomena has been manifest. One of the first detailed examples is given by Fermat's principle relative to the economy of time in the transmission of light. It was next recognised that the general equations of mechanics correspond to a problem of minimum, or, more exactly, to a problem of variation; and thus were derived the principle of virtual velocities, and, later, Hamilton's principle, and the principle of least action. A large number of problems then appear to correspond to the minima of certain definite integrals. This was a very important advance, for the existence of a minimum in many cases could be regarded as evident, and therefore the proof of the existence of a solution was effected. This method of reasoning rendered vast services. The greatest geometers—Gauss, in the problem of the distribution of an attracting mass according to a given potential; Riemann, in his .theory of Abelian functions—were satisfied with it. Our attention has been recently drawn to the danger of proofs of this kind; it may be that the minima are simply limits, and that they cannot be effectively attained by real functions possessing the necessary properties of continuity. We are therefore no longer content with the plausibilities which are afforded by the form of reasoning which was classical for so long. Whether we proceed indirectly, or whether we try to give a direct proof of the existence of a function corresponding to the minimum, our way is long and arduous. But it will be none the less useful in every case to connect a question in mechanics or mathematical physics with a problem of minimum. In this there is from the outset a fertile source of analytical transformations, and besides, in the very processes of the investigation of variations, useful indications may appear relative to the conditions for the limits. A fine example has been given of this by Kirchoff, in his delicate investigation of the conditions for the limits of the equilibrium of flexure of plates.

VI.

I have been led to extend my remarks in particular on partial differential equations. Examples selected from theoretical mechanics and from celestial mechanics would readily show the part that is played by ordinary differential equations in the progress of these sciences, the history of which, as we have seen, has been so closely connected with that of analysis. When the hope of integrating by means of simple functions was abandoned, we were compelled to find developments enabling us to follow up a phenomenon as long as possible, or at any rate to obtain information as to its qualitative aspect. Practically, methods of approximation play an extremely important part in mathematics, and it is by their means that the most advanced parts of theoretical arithmetic come into connection with the applied sciences. As for series, the very existence-proofs of the integrals furnish them from the outset. Thus Cauchy's first method gives developments which are convergent so long as the integrals and the differential coefficients remain continuous. When circumstances enable us to foresee that this will always be the case, the developments obtained are always convergent. In the problem of the n bodies we can in this manner obtain some developments which are valid as long as there are no impacts. If the bodies, instead of attracting, repel one another, this fact need not dismay us, and we obtain developments which are valid indefinitely. Unfortunately, as Fresnel said one day to Laplace, "Nature is not disturbed by analytical difficulties," and the celestial bodies attract and do not repel. In the same way one would be tempted at times to go even farther than the great physicist, and to say that nature has sown the path of the analyst with difficulties. Thus to take another example, if we are given a system of differential equations of the first order, we can generally decide if the general solution is stable about a point or not, and we can find developments in series valid for stable solutions; the sole condition is that certain inequalities must be verified. But if we apply these results to the equations of dynamics in the discussion of stability we find ourselves exactly in the one particular unfavourable case. In general, even here it is not possible to come to a conclusion as to stability. In the case of a force-function having a maximum, the classical, though indirect, method of reasoning establishes the stability which cannot be deduced from any development valid for every value of the time. Do not let us be dismayed by these difficulties. They will be the source of future progress. Such are also the difficulties that, in spite of the attention that has been paid to them, meet us in the equations of celestial mechanics. From the days of Newton astronomers have drawn from them, by means of practically convergent series and skilfully devised approximations, almost all that is necessary for the prediction of the movements of the heavenly bodies. Analysts would ask more, but little hope is left of reaching the integration by means of simple functions or of developments always convergent. What the admirable researches of modern times have taught them most is the immense difficulty of the problem. A new way has, however, been opened by the study of particular solutions, such as the periodic solutions and the asymptotic solutions which have already been utilised. It is not so much perhaps on account of practical requirements as for the fear of confessing itself vanquished that analysis would never resign itself to abandon without a decisive victory a subject in which it has met with so many brilliant triumphs; besides, what finer field could be found by the budding or rejuvenated theories of the modern doctrine of functions in which to try their strength than in this classical problem of the n bodies?

It is a delight to the analyst in his applications of equations that he can integrate to meet with known functions, with transcendents already classed. Such meetings are unfortunately rare. The problem of the pendulum, the classical cases of the motion of a solid body around a fixed point, are simple instances in which integration has been effected by means of elliptic functions. It would be also extremely interesting to meet with a question in mechanics which might be the starting point of an important discovery in the theory of functions, such as the discovery of a new transcendental enjoying some remarkable property. I should find some difficulty in giving an instance without having to go back to the pendulum and the origin of the theory of elliptic functions. The interpenetration between theory and application is here much less than was the case in questions of mathematical physics mentioned above. Thus is it explained how in the last 40 years researches on the ordinary differential equations connected with analytical functions have in a large measure an entirely abstract theoretical character. Pure theory is here notably in advance. I have had occasion to observe that it is a good thing that it is so, but here the question is evidently one of proportion, and we may hope to see the old problems profit by the progress accomplished in pure theory. There is no difficulty in giving examples, and I will merely remind you of those linear differential equations involving arbitrary parameters, the singular values of which are roots of transcendental integral functions, and one of which in particular brings into correspondence the successive harmonics of vibrating membranes with the poles of a meromorphic function.

It happens also that the theory may be an element of classification by leading us to seek for the conditions under which the solution is of a given type, as, for example, when the integral is a uniform function. There have been, and there will be again, many interesting discoveries along this line. The case of the motion of a heavy solid body treated by Mme. de Kowalewski, in which Abelian functions were utilised, has already become a classical instance.

VII.

While studying the reciprocal relations of analysis, mechanics and mathematical physics, we have on our way more than once come into contact with the infinitesimal geometry to which so many celebrated problems are due. In many difficult questions the happy combination of calculus and synthetic reasoning has realised considerable progress, as may be seen in the theories of applicable surfaces and of triple orthogonal systems. There is another part of geometry which plays a great rôle in certain analytical researches—I mean the geometry of situation or *analysis situs.* We know how Riemann has from this point of view made a complete study of the continuum of two dimensions, on which he bases his theory of algebraical functions and their integrals. As the number of dimensions increases, the question of *analysis situs* necessarily become complicated. Geometrical intuition ceases, and the study becomes purely analytical, the mind being guided solely by analogies which may be misleading and must be closely investigated. The theory of algebraical functions of two variables, which carries us into the space of four dimensions, without deriving from *analysis situs* such fruitful aid as has been derived by the theory of the functions of one variable, nevertheless owes to it some useful orientations. Again, there is another order of questions in which the geometry of situation intervenes. In the study of curves traced on a surface and defined by differential equations, the connectivity of this surface plays an important part, and this is notably the case in dealing with geodesic lines. The question of connectivity also occurred long since in analysis, when the study of electric currents and magnetism led to non-uniform potentials. In a more general manner certain multiform integrals of some partial differential equations are met with in difficult theories such as that of diffraction, and there is room for various researches in this direction.

From a different point of view I may here recall the relations of algebraical analysis with geometry, which are so elegantly manifested in the theory of groups of finite order.

A regular polyhedron, such as an icosahedron, is on the one hand the solid familiar to us all. Also, to the analyst, it is a group of finite order corresponding to the different ways in which the polyhedron can be made to coincide with itself. The search for all the types of groups of movements of finite order interests not only geometers but also crystallographers. It amounts essentially to the study of groups of ternary linear substitutions with determinant ±1, and leads to the crystallographers' 32 classes of symmetry for the complex particle of a Bravais lattice. The packing of similar polyhedra in such a way as to fill space completely exhausts every possibility that may arise in the investigation of the structure of crystals. Ever since the idea of the group was introduced into algebra by Galois it has been considerably developed in various ways, so that we now meet it in every branch of mathematics. In its applications especially it appears to us as an admirable instrument of classification. Whether it be a question of substitution groups or of Sophus Lie's transformation groups, whether it be a question of algebraical or of differential equations, this doctrine, so comprehensive in its scope, enables us to deal with the degree of difficulty of the problems treated, and teaches us to utilise the special circumstances that are presented. On this ground it must prove as valuable in mechanics and mathematical physics as in pure analysis. The degree of development of mechanics and physics has enabled us to give to almost all their theories a mathematical form. Certain hypotheses and a knowledge of the elementary laws have led to differential relations which constitute the final form under which these theories, at any rate for a time, are fixed. Little by little they have seen their field enlarge with the principles of thermodynamics. Chemistry, too, tends nowadays to assume a mathematical form. I shall only refer as a case in point to the celebrated memoir of Gibbs on the equilibrium of chemical systems, so analytical in its character, in which chemists could not without an effort recognise in their algebraical garb laws of considerable importance. It seems as if chemistry has nowadays emerged from the pre-mathematical method with which every science begins, and that the day must come in which will be systematised theories of vast importance and magnitude, analogous to those of our present mathematical physics, but much wider in scope and comprising all physico-chemical phenomena. It would be premature to ask if analysis will find in their developments the source of further progress. We cannot even conjecture the analytical types with which we shall be confronted. I have repeatedly mentioned the differential equations which regulate phenomena. Will this always be the final form into which a theory is crystallised? On this point, indeed, I know nothing; but we must remember, however, that several hypotheses have been made of a more or less experimental nature. Among them there is one which has been called the principle of *non-heredity*, which postulates that the future of a system depends only on its present state and on its state at an infinitely near moment, or more briefly, that accelerations depend only on positions and velocities. We know that in certain cases this hypothesis is not admissible, at least with the magnitudes directly considered; and sometimes an unjustifiable advantage has been taken of this principle, as, for instance, in the "memory" of matter which recalls its past; and authors have written in terms of deep emotion about the "life" of a fragment of steel. An attempt has been made to construct a theory of these phenomena in which the distant past seems to intervene. Of this I need not speak here. An analyst may think that in cases so complex we must abandon the form of differential equations, and resign ourselves to the consideration of *functional equations*,[1] in which will figure definite integrals bearing witness to an heredity of some sort. When we see the interest which at the present moment attaches to functional equations, one might almost consider it to be due to a presentiment of our future requirements.

VIII.

After having spoken of non-heredity, I scarcely dare touch upon the application of analysis to biology. No doubt it will be some time before we form the functional equations of biological phenomena of a type analogous to those of which I have just spoken.[2] Attempts made so far are not very ambitious. However, we are endeavouring to leave the purely qualitative field and to introduce quantitative measurements. In the question of the variation of certain characters we devote our attention to the statistics of measurements, and represent the results by curves of frequency. The modifications of these curves with successive generations, their decompositions into distinct curves, may give us the measure of the stability of a species or of the rapidity of mutations, and we know the interest that attaches to these questions in recent botanical research (*e.g.* H. de Vries' researches on *Œnanthera*). In all this there are so many parameters that we are led to ask if the infinitesimal method itself may be of any service. Certain laws of a simple arithmetical character, such as those of Mendel, sometimes give us fresh confidence in the old aphorism which I quoted at the beginning of this address, that everything is explained by numbers; but in spite of our legitimate hopes, it is clear that, as a whole, biology is still far from entering upon a really mathematical period.

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[1] I have read M. Fredholm's remarkable paper in the *Acta Mathematica*, t. xxvii., and what was subsequently published in Germany.

[2] In an article on *Lamarck's Principle and the Heredity of Somatic Modifications*, M. Giard thus speaks of heredity: "It is an integral, the sum of the variations produced on each anterior generation by the primary factors of evolution." V. *Controverses transformistes*, p. 135.

This is not the case, according to certain economists, with political economy. After Cournot, the Lausanne school made an extremely interesting effort to introduce mathematical analysis into political economy. Under certain hypotheses, which suit at any rate the limiting cases, we find in learned treatises the equation between the quantities of goods and their prices, reminding us of the equation of virtual velocities in mechanics. This is the equation of economical equilibrium. A function of the quantities plays in this theory an essential part, recalling that of the potential function. Moreover, the most authoritative spokesmen of the school insist on the analogy between economical and mechanical phenomena. "Just as theoretical mechanics considers material points, so," says one of these experts, "pure economy considers the *homo œconomicus*." Naturally, we also find here the analogues of Lagrange's equations, that indispensable matrix of all mechanics. But while we admire these bold investigations, we cannot help feeling that the authors may have neglected certain "hidden masses," as Helmholtz and Hertz would say. But though that may be the case, in these theories there is a curious application of mathematics which, at least in some well circumscribed cases, has already rendered great services.[1]

Gentlemen, I now bring to a close this summary of some of the applications of analysis, and the reflections which have been momentarily suggested by it. It is far from complete; for instance, I have made no mention of the calculus of probabilities, which requires so much subtlety of treatment, the refinements of which Pascal refused to explain to the Chevalier de Méré because he was not a geometer. Its practical utility is of the first rank; its theoretical interest has always been considerable; and it is still greater in the present day, thanks to the importance assumed by the researches which Maxwell called *statistical*, and which tend to exhibit mechanics in quite a new light.[2]

I hope, however, that I have shown in my sketch the origin and the reason of the intimate ties uniting analysis to geometry and physics, and, more generally, to every science bearing upon numerically measurable magnitudes. The reciprocal influence of analysis and physical theories has been in this respect peculiarly instructive. What has the future in store? More difficult problems, corresponding to approximation of a higher order, will bring in complications which we can only vaguely fore-cast by speaking, as I did just now, of functional equations systematically replacing our present differential equations, or again, of integrations of equations infinite in number and involving an infinity of unknown functions. But even should that come to pass, mathematical analysis will always remain that language which, as Fourier said, "has no symbols to express confused ideas," a language endowed with a wonderful power of transformation, and capable of condensing within its formulas an immense number of results. EMILE PICARD.

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[1] *v. La méthode mathématique en Économie politique*, by E. Bouvier, and *Petit Traité d'Économie politique mathématique*, by H. Laurent.

[2] Cf. *Elementary Principles in Statistical Mechanics*, by J. Willard Gibbs, and *Leçons sur la théorie des gaz*, by L. Boltzmann.

REVIEWS.

Kummer's Quartic Surface. By R. W. H. T. HUDSON, M.A., D.Sc. Pp. xii, 222. (Cambridge: at the University Press, 1905.)

The appearance of this book is likely to add, if that be possible, to the general regret aroused by the premature death of the author. The idea of writing it was an excellent one, and has been carried out with marked success, considering the difficulty of choosing and arranging the material. For the study of Kummer's Surface involves a combination of a great variety of analytical methods; groups, configurations, line geometry, non-Euclidean geometry, and theta-functions all play an important part; and in a case of this kind it is not easy to decide how much to take for granted as already familiar to the reader.

The most striking feature of the treatise is the way in which, by a style which is concise without being obscure to a really attentive student, the author has contrived to give the necessary amount of these auxiliary theories so as to be intelligible without reference to other text-books. As an example we may take the last chapter, which deals with Singular Kummer Surfaces. True, this is only a sketch, which requires supplementing by reading the papers of Humbert and others; but it does show very well the *raison d'être* of the transformation of theta-functions, and the effect of the occurrence of singular moduli. And so in other chapters the treatment is broad and suggestive; the main points of the subject are brought out in a way likely to arouse interest, and lead to further research on the part of the reader. This is a pre-eminent merit in a mathematical treatise; and it must have also belonged to the lectures on Kummer's Surface which Hudson gave while preparing his book.

An illustration of Hudson's clearness of thought and expression is given by what he says about geometry of four dimensions. "By 'geometry of four dimensions' is to be understood a method of reasoning about sets of numbers and equations, in which the principles of elementary algebra are clothed in a language analogous to that of ordinary geometry. Although we cannot bring our intuition to bear directly upon four-dimensional configurations, we can do so indirectly by creating an artificial intuition based on analogy." It must be understood, of course, that by "four-dimensional configurations" are meant configurations in a four-dimensional space of points—this is clear enough from the context; ordinary space with lines or spheres as elements does afford an intuition of a kind, though not analogous in the same immediate way as ordinary solid geometry is to the ordinary geometry of the plane. But with this reservation the passage quoted truly represents the proper way of regarding geometry of four or more dimensions. The power which this "artificial intuition" gives to those who cultivate it is really remarkable; an illustration has been recently given by Mr. Richmond, whose way of obtaining the Kummer (16, 6) configuration is summarised on pp. 129-30 of Hudson's book.

Naturally, Klein's applications of line geometry receive considerable attention. In many respects the fundamental properties of Kummer's Surface, and of its degenerate forms, are closely connected with a family of quadratic complexes; and this is one direction in which further study seems likely to be fruitful. Another is in connection with the application of theta-functions. It is proved by means of them (pp. 184-5; cf. pp. 138-40) that a surface can be found to touch Kummer's Surface all along any given algebraic curve lying thereon, and have no further intersection with the surface. Now this is a purely algebraic theorem, and has no intrinsic connection with theta-functions; it ought, therefore, to be possible to prove it without their aid. Hudson is possibly right in supposing that such a proof would be long and complicated (p. 185); but to provide it would be a real step in advance, and could hardly fail to bring out some characteristic properties of the surface.

No object would be gained by analysing the contents of this treatise in detail: it will be enough to say that, besides the general Kummer Surface, the degenerate forms receive due attention; that conscientious reference is made to the original sources; and that a photograph of a plaster model of the general surface, prefixed to the volume, will greatly help the reader to understand the account given of its geometrical properties. The duty of seeing the later part of the work through the press has been performed by Dr. H. F. Baker and Mr. H. Bateman; a prefatory note by the former contains a brief account of Hudson's career, and a sympathetic appreciation of the merits of this treatise. G. B. MATHEWS.

Manual of Quaternions. By Professor C. J. JOLY. (Macmillan & Co., 1905)

Tait has said somewhere that Hamilton first invented the quaternion and then discovered it. The invention consisted in the conception of the three imaginaries *i*, *j*, *k*, with their special laws of combination, and in the construction of an associative algebra with four fundamental units. The quaternion was then discovered to be a complex number representing the ratio of two vectors or directed lines in space. Hamilton was the first who clearly recognised the value of the associative law; and in spite of many imitations his system remains the only tridimensional system of vector analysis governed by this law.

The mathematical world has, however, been slow to recognise the essential merits of Hamilton's calculus. A vast deal of ingenuity and time has been spent by some in finding new notations of the quantities and operators peculiar to quaternions; some have even been beguiled by these notations into a rediscovery of theorems as old as Hamilton's

Lectures. The curious thing is that it does not appear that any really fresh mathematical truth has been brought to light by the inventors and users of vector notations intended to supersede Hamilton's original system. Here, of course, we refer only to tridimensional applications.

It is very refreshing then to open the pages of a book whose author, boldly accepting the form of the calculus as Hamilton developed it, proceeds to unfold its beauties and strength with all the skill of a practised hand. The book begins with a very brief chapter on the addition and subtraction of vectors, a part which necessarily occupied a considerable section of Hamilton's and Tait's treatises. The vector conception has now crept into our elementary books, and in due course will probably become a conspicuous feature even of our most elementary geometries. Graphical methods have within recent years transformed our teaching of algebra; and the vector as a geometrical entity is essentially graphical.

Professor Joly wisely assumes that for students who have made some progress in mathematics the law of vector addition calls for little elaboration. He devotes five pages to it and then plunges into quaternions proper. He takes what he believes to be the "shortest and simplest route." The student he says "cannot be expected to undertake the study of quaternions in the hope of being rewarded by the beauty of the ideas and by the elegance of the analysis. And for his sake, though with reluctance I must confess, I have abandoned Hamilton's methods of establishing the laws of quaternions." Perhaps so, but it is a poor student who despises logical development, beauty of idea, elegance of method, and is content with a "working knowledge of the calculus." We confess to an uneasy feeling that principle has here been sacrificed to expediency. There is a suggestion of "tumbling over the wall" and not coming "in at the Gate which standeth at the beginning of the way." We have often wondered how many students study Clifford's Dynamic for the sake of learning dynamics. Possibly none; and probably as few have *learned* quaternions from that book. Although there is some initial similarity between Professor Joly's method and the tentative nibbling at quaternions which characterises Clifford's book, there is almost immediately a vast divergence. The true quaternion is introduced on page 9, and it dominates the whole treatise. This is as it ought to be. It is possible, as Heaviside has shown, to use effectively much of the notation of quaternions without explicit use of the quaternion itself; but sooner or later the lack of it will be felt. The student should never lose sight of the fact that $S\alpha\beta$ and $V\alpha\beta$ together form by addition a quantity which, however it may operate on or be taken in conjunction with a like quantity, gives rise to a quantity of the same analytical nature. This is the central doctrine of the quaternion calculus.

To give any complete idea of all that Professor Joly's volume contains would be practically to reproduce his table of contents. Beginning with the simpler applications to trigonometry and to the geometry of plane and sphere, he quickly passes into the peculiarities of quaternion differentiation and into the exquisite theory of the linear vector function (or matrix), after which he is ready for all kinds of applications in geometry of curves and surfaces, and in kinematics and dynamics. The reader cannot fail to be impressed with the directness of the method in all these applications, especially if he is familiar with the ordinary modes of attack. In virtue of this directness of attack and the extraordinary conciseness of notation more detail can be packed into one quaternion page than into three or four pages of ordinary analysis. By what other method, for example, could systematic discussions of line, surface and volume integrals, of spherical harmonics, heterogeneous strain, elastic vibrations, and electro-magnetic theory be given in less than fifty pages? In the variety of the mathematical and physical subjects taken up there are only two other books which can compare with Professor Joly's *Manual*, and these are Hamilton's *Elements* and Tait's *Treatise*.

The greater part of the book is necessarily a development of much that is to be found in the pages of Hamilton, Tait, and M'Aulay; but Professor Joly has a characteristic style of his own, more nearly akin to Hamilton's than to Tait's. In the last two chapters especially are the author's additions more in evidence. These are on Projective Geometry and Hyperspace. The former is based upon a new interpretation of the quaternion; and in the latter Professor Joly gives a sketch of the properties of associative algebras applicable to *n*-dimensional space.

Professor Joly has certainly succeeded in his aim of providing the student with a *working* book. He takes excursions into many fields of mathematics pure and applied, and the treatment is not superficial. Important applications are worked out in detail; and numerous examples are given by which the student may test his progress. Let the reader accept on trust the initial assumptions and developments, and work earnestly through the succeeding chapters. He will come out in the end a practised quaternionist. C. G. KNOTT.

Leçons sur les fonctions de variables réelles, par E BOREL; **Leçons sur les fonctions discontinues**, par RENÉ BAIRE; **Le calcul des residus et ses applications à la théorie des fonctions**. par E. LINDELÖF. (Paris, Gauthier-Villars, 1905, 3 f. 50 c. each.)

M. Borel's book is the sixth of his series of monographs on the theory of functions, of which the first appeared as recently as 1898. M. Borel is only human, and by now he has decided to leave to his pupils the work of preparing his lectures for publication. It cannot be said that this method has proved in every case an unqualified success. The first few volumes, prepared by M. Borel himself, and particularly the admirable *Leçons sur les fonctions entières*, were remarkable alike for their originality, for the judgment shewn in the selection of material, and for the lucidity and proportion of the exposition. This high standard has not been maintained in all of the later volumes, some of which have been rather scrappy, and have given the impression of hasty, and at times perfunctory composition. In these respects, however, the present volume is an improvement upon its immediate predecessors, M. Maurice Fréchet having performed his task unusually well. But I cannot help thinking that M. Borel would be fortunate if he could find the time to write his books himself.

The principal problem with which M. Borel deals in this volume is that of the representation of functions by means of series of polynomials, a form of representation the importance of which was first shown by Weierstrass's well-known theorem that every continuous function of a real variable can be expanded in such a series. M. Borel confines himself to functions of real variables, reserving the complex theory for the next volume of his series; and for the most part he is concerned with continuous functions only, the short chapter on the representation of discontinuous functions containing little more than a discussion of the comparatively simple case in which the aggregate of points of discontinuity is enumerable, and a reference to the results obtained by M. Baire. The consequence is that the book is rather disconnected; for the first two chapters, which deal with the theory of aggregates and continuity and discontinuity in general, contain a good deal which, though very interesting in itself, is really not required for M. Borel's purpose. The central chapters (3 and 4) are excellent,[1] and the long note added by M. Painlevé is perhaps the most interesting feature of the book, though its natural place would not be in this volume, but in the next one.

There is one criticism which will occur to every regular reader of this series, in which M. Borel has now enlisted the collaboration of a number of other eminent mathematicians. There is an amount of repetition which a judicious general editor should be able to diminish; and that the plan of the series is open to this criticism M. Borel, to judge from his remarks in the preface, appears to recognise: 'Il a paru préférable d'admettre parfois quelques brèves redites plutôt que de renoncer à l'indépendance des Volumes de la Collection, chacun d'eux devant pouvoir être lu isolément par un lecteur ayant des connaissances générales d'Analyse. Sans ce principe d'indépendance, on aurait eu tous les inconvénients d'un grand Traité, sans en avoir les avantages.'

In principle, no doubt M. Borel is right. Each author should be asked to deal with some definite question, and he should have full liberty to preface his discussion of it with a general account of those modern developments of analysis which are necessary for his purpose, and with which a reader who has not read the other volumes of the series cannot be expected to be familiar. But he should be very careful to make this general account as short as is consistent with clearness, and to limit it strictly to results which will afterwards be required. This is the course adopted by M. Baire, who, if he is at times a little diffuse, is careful not to encumber his book with unnecessary matter. M. Borel has not set his colleagues so good an example. Why, for example, should he think it necessary to introduce a few pages concerning M. Lebesgue's generalisation of the notion of the definite integral? Most interesting and most important this generalisation certainly is; but it has already been expounded by M. Lebesgue himself in an earlier volume of the series, and no allusion whatever is made to it throughout the remainder of the book.

It will not be necessary to say much about the volume contributed by M. Baire. It is in substance a popular edition of his remarkable memoir, *Sur les fonctions de variables réelles*, published as a thesis in 1899, and afterwards in the *Annali di Matematica*. A good deal of introductory matter has been added, and the argument has been simplified and condensed. The problem of finding the *necessary and sufficient* conditions that a function whose points of discontinuity are given should be capable of representation as the sum of a series of continuous functions is one which most mathematicians would have regarded as hopeless if M. Baire had not completely solved it; and that M. Baire's researches should be made more accessible to the ordinary reader was much to be desired. But it is difficult to resist the impression that these two volumes might well have been condensed into one. M. Baire's results are particularly interesting when applied to differential coefficients of continuous functions. If $f(x)$ is continuous, so is

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[1] The argument of p. 66 has become inverted in some curious way. The function is '*more* continuous' when $\phi(\epsilon)$ decreases *less* quickly.

$$\frac{f(x+h)-f(x)}{h}$$

considered as a function of x. Now, if the differential coefficient exists for every value of x under consideration, and we denote by h_1, h_2, h_3 ... a series of positive quantities whose limit is zero,

$$f'(x)=\lim_{n=\infty}.\frac{f(x+h_n)-f(x)}{h_n},$$

and is therefore representable as the limit of a sequence of continuous functions, or (what is the same thing) the sum of a series of such functions. M. Baire's results, therefore, give us much important information as regards the possible discontinuities of the differential coefficients of continuous functions.

For M. Lindelöf's *Calcul des Résidus* I have nothing but praise. The applications of Cauchy's "calculus" to the theory of functions, and in particular to the summation of series and the theory of analytic continuation, are of the most far-reaching character, and, so far as I know, no one before M. Lindelöf has attempted to give a systematic account of them. Laurent, it is true, published in 1865 a *Théorie des Residus*, which hardly deserved to have been so soon forgotten, but the applications of the theory have multiplied ten times since then. Some new account was urgently necessary, and M. Lindelöf has given us exactly what was wanted. One admirable feature of his book is the thoroughness and exactitude of his historical references, especially to the writings of Cauchy. M. Lindelöf is one of the few mathematicians who have found life long enough to make 'une étude détaillée' of Cauchy's works.

M. Lindelöf very wisely does not trouble himself with all the difficulties as to the *minimum* of assumption required to establish Cauchy's theory, which centre round Goursat's proof of Cauchy's theorem. These difficulties, of course, have absolutely no bearing on the applications with which M. Lindelöf is chiefly concerned. Assuming the continuity of the differential coefficient, he proceeds to show that any analytic function $f(x)$ is itself the differential coefficient of an analytic function $F(x)$ determinate save for an additive constant, a conclusion from which, combined with the definition of the definite integral along a curvilinear path, Cauchy's theorem immediately follows. There is certainly a great deal to be said for presenting the proof of the theorem in this way.

A short but clear account follows of some familiar applications of the formula

$$f(x)=\frac{1}{2\pi i}\int\frac{f(t)}{t-x}dt$$

and of some others which are not so familiar, such as occur in the proof of Jensen's theorem, the theory of the Bernoullian and Eulerian functions, the factorisation of such functions as $\sin x - ax\cos x$, and the transformation of slowly convergent series. In Chapter III. he comes to the ground which he has made particularly his own. He proves a whole series of general formulae, of which the formula of Plana and Abel,

$$\Sigma f(x)= -\tfrac{1}{2}f(x)+\int^{x}f(x)dx+\frac{1}{i}\int_{0}^{\infty}\frac{f(x+it)-f(x-it)}{e^{2\pi t}-1}dt,$$

was historically the first. The applications of these formulae to different regions of analysis—Gauss's sums, the Zeta and Gamma functions, the continuation of power series, and the asymptotic behaviour of integral functions—are so numerous that it is impossible to enumerate them here. I will only cite the beautiful formula which defines the behaviour of the function

$$F(x,s)=\frac{x}{1^s}+\frac{x^2}{2^s}+\frac{x^3}{3^s}+\dots$$

near $x=1$, viz.,

$$F(x,s)=\Gamma(1-s)\left(\log\frac{1}{x}\right)^{s-1}+\sum_{0}^{\infty}\zeta(s-\nu)\frac{(\log x)^{\nu}}{\nu!},$$

which solves completely a problem which has exercised the ingenuity of many mathematicians.

M. Borel's series of monographs is almost indispensable for anyone who is engaged in research in the theory of functions, and with the possible exception of M. Borel's own first two volumes, M. Lindelöf's contribution certainly seems to me the best. G. H. HARDY.

Elements of the Kinematics of a Point and the Rational Mechanics of a Particle. By G. O. JAMES, Ph.D., Instructor in Mathematics, Washington University, St. Louis. New York: Wiley & Sons. London: Chapman & Hall. 1905. Pp. 176. $1.

Presupposing some familiarity with elementary experimental mechanics, and a knowledge of the calculus, the author has written a sketch of the "dynamics of a particle," paying special attention to the effect of the earth's own motion on the motion of a particle on the earth's surface.

The entire absence of examples will appear to many as a defect, but otherwise the book may prove helpful in enabling a student to generalise some of the limited concepts gained by previous experimental work, and to grasp the difficulties of abstract dynamics. C. S. J.

Integral Calculus for Beginners. By ALFRED LODGE, M.A., Mathematical Master at Charterhouse, formerly Professor of Pure Mathematics at the R.I.E. College, Coopers Hill. London: George Bell & Sons. 1905. Pp. 203. 4s. 6d.

This is a companion volume to Professor Lodge's *Differential Calculus for Beginners*, which was reviewed in the *Gazette* [vol. ii., p. 213].

In that volume the student was prepared to practise retracing his steps, and thus, without the use of the integral notation, to perform the operation of integration or anti-differentiation in simple cases. Hence the author is in a position to commence this volume by exhibiting an integral as the limit of a sum; and that no time is wasted in getting to business is evidenced by the fact that the centre of gravity of a parabolic area is worked out at p. 9. The standard methods of integration are clearly explained and illustrated in the first five chapters.

The most novel feature of the book is perhaps the seventh chapter dealing with approximate methods of integration. Here, after the well-known rules of Simpson and Weddle, approximate formulae, recently devised by Mr. R. W. K. Edwards and Professor Lodge himself, are given, for dealing with the case in which the curvilinear boundary of a required area cuts the axis at right angles; a case for which, as is well known, rules of the Simpson type are not well fitted.

Interesting approximate formulae for the elliptic integrals are also given.

A chapter on Moments of Inertia is very welcome, and the book concludes with a chapter on the Gamma functions and with chapters on the differential equations, other than partial, of most frequent occurrence.

The suggestion may be submitted for consideration in a future edition that, while doubtless the theory of Amster's planimeter is too difficult for a first book on the Integral Calculus, yet some of the earlier instruments described in Professor Henrici's British Association report (1894) perform the process of summing up *ydx* in an obvious manner; and the Integraphs of Professors Boys and Abdank Abakanowicz are also exceedingly interesting concrete embodiments of Integration, viewed as the converse of differentiation.

Professor Lodge's book is likely to maintain the position which his book on the Differential Calculus has won. C. S. JACKSON.

Mechanics. A school course. By W. D. EGGAR, M.A. London: Edward Arnold. 1905. Pp. viii, 288. 3s. 6d.

Mr. Eggar's *Practical Exercises in Geometry* was one of the first of the numerous books which have endeavoured to embody, in a concrete form, the ideas of the "New Geometry."

He has now produced a pioneer book on mechanics, in which simple yet striking experiments and abundant illustrations from familiar objects will compel a student to look on things as they are, and will dissuade him from using terms of art without a notion of their meaning.

The work commences with kinetics, but is so arranged that teachers who prefer to begin with statics can do so. A full account of Galileo's experiments on falling bodies introduces the beginner to a classical example of scientific investigation.

The book is admirably written, and covers more ground than, from its moderate size and attractive print, one would at first imagine; nor has clearness been sacrificed for the sake of brevity, with perhaps one exception in Article 126, dealing with "centrifugal force." Here a diagram, in which the "centrifugal force" is shewn ranking with the real forces, may cause confusion, which the brief statement in the text may not fully remove.

The diagrams are remarkably neat and clear, and there are examples which, as mere algebraical and trigonometrical conundrums have been rigorously excluded, are really examples in mechanics.

On page 13 should not N. 53° E. be E. 53° N.? C. S. JACKSON.

Stereoscopic Views of Solid Geometry Figures. D. C. Heath & Co., Boston, U.S.A.

This set of slides, 93 in number, has been specially designed with reference to the 'Essentials of Solid Geometry' by Prof. W. Wells, a book which has been favourably noticed for the excellence of its text and the diagrams with which it is illustrated. The slides seem to be just what is wanted by a pupil temporarily puzzled as to the position and arrangement of the lines represented by those in the plane diagram. There is a future before the stereoscope in this department of education. Besides the direct use to a beginner of Bk. XI or an equivalent course, we believe that it is likely to act as a powerful incentive to a student to learn enough perspective to enable him to draw his own 'stereographs.'

E. M. LANGLEY.

MATHEMATICAL NOTES.

173. [K. 13. a.] The *Remarque Minuscule* (Note 167, *Gazette*, May 1905, p. 176) has also been made by Mr. Steggall (*Proc. Edin. Math. Soc.*, vol. x, 1892.

N. QUINT ('s-Gravenhage).

174. [D. 2. a.]. *The criterion as to a sequence tending to a limit.*

If $s_1, s_2, s_3 \dots$ be the sequence of real or complex quantities, it does or does not tend to a limit according as it is or is not true that with every positive ϵ, assigned as small as we please, there can be associated a number n such that, for every p, $|s_n - s_{n+p}| < \epsilon$. The ordinary proof that the existence of a limit necessitates this inequality presents no difficulty, but proofs of the converse appear to me needlessly encumbered. The following, in two stages, seems conclusive.

(1) *If the sequence tends to no limit, a positive* ϵ' *can be taken so small that, whatever quantity* s *be, and whatever number* n *be,* $|s - s_{n+p}| \nless \epsilon'$ *for some* p.

For suppose this is not so. Then with any particular ϵ' goes at least one value of s such that, for some n and every p, $|s - s_{n+p}| < \epsilon'$. A value of s with this property for one ϵ' has it for every greater ϵ'. As ϵ' is diminished, it follows that no new s with the property can arise, but the range of values of s with the property may well diminish. We are supposing, however, that ϵ' cannot be so diminished, remaining assignable and positive, that all values of s cease to have the property. Let s' be one which retains it. Then we are told that, however small ϵ' be assigned, there is always some n such that, for every p, $|s' - s_{n+p}| < \epsilon'$; and this means that the sequence tends to the limit s', so that the supposition made is untenable.

(2) *If, however small* ϵ *be assigned, there is always some corresponding* n *such that, for every* p, $|s_n - s_{n+p}| < \epsilon$, *the sequence tends to a limit.*

For suppose the contrary. Then by (1) there is an assignable ϵ' such that with every value s and every m goes some q such that $|s - s_{m+q}| \nless \epsilon'$. Also our datum assures us that with the ϵ' in question goes some n such that, for every p, $|s_n - s_{n+p}| < \epsilon'$. In these two inequalities s and m may be chosen at will. Give them the values s_n and n, which are definite. Moreover, p may be chosen at will. Give it the value q, which is made definite by the definite choices of s and m. We thus get two inequalities,

$$|s_n - s_{n+q}| \nless \epsilon' \text{ and } |s_n - s_{n+q}| < \epsilon',$$

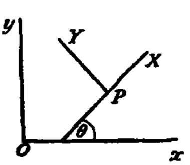
which are inconsistent. We have then supposed an impossibility.

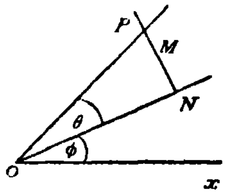
E. B. Elliott.

175. [K. 20. a.]. *The Addition Formulae for Cosine and Sine.*

If PX makes an angle θ with Ox, then the projection of PX on Ox is $PX \cos\theta$. Hence if Oy, PY are obtained from Ox, PX by a turn through a right angle in the positive direction, it follows that the projection of PY on Ox is $PY\cos\left(\theta + \frac{\pi}{2}\right)$, and that the projection of PX on Oy is $PX\cos\left(\theta - \frac{\pi}{2}\right)$.

Now let ON make an angle ϕ with Ox and OP an angle θ with ON. Let NM be the direction obtained from ON by a turn through a right angle in the positive direction. Let NM cut OP in P and take OP as the unit of length. Then the projection of OP on Ox is $\cos(\theta+\phi)$. But this projection is also the sum of the projections on Ox of ON, NP, that is of $\cos\theta$, $\cos\left(\theta - \frac{\pi}{2}\right)$. Now the projecting factors are $\cos\phi$, $\cos\left(\phi + \frac{\pi}{2}\right)$. Hence we have





$$\cos(\theta+\phi) = \cos\theta\cos\phi + \cos\left(\theta - \frac{\pi}{2}\right)\cos\left(\phi + \frac{\pi}{2}\right)$$
$$= \cos\theta\cos\phi - \sin\theta\sin\phi.$$

The proof applies for angles of any size or sign.

Replacing ϕ by $\phi - \frac{\pi}{2}$, we get the addition formula for sine, and then, in the two addition formulae, replacing ϕ by $-\phi$, we get the formulae for the sine and cosine of $\theta - \phi$.

E. J. Nanson.

176. [D. 6. b.]. *The Fundamental Exponential Limit.*

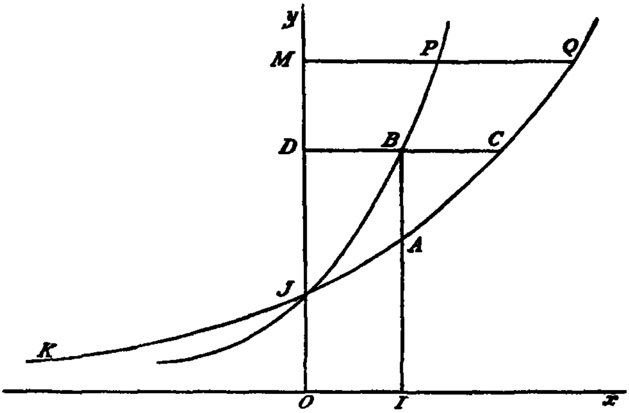
Let the curve KJQ be the graph of $y = a^x$ and let $OI = OJ = 1$, so that $IA = a$. In IA or IA produced take any point B where $IB = b$ and through B draw DBC parallel to Ox cutting Oy in D and the graph in C. Then if $DC = c$, we have $b = a^c$, and therefore $b^x = a^{cx}$. Hence if through any point P on the graph JBP of $y = b^x$ MPQ is drawn parallel to Ox cutting Oy in M and the graph of $y = a^x$ in Q, then $MQ = c \,.\, MP$.

Thus from the graph of any one exponential a^x that of any other exponential b^x can be deduced by cutting all the ordinates to y in the proper ratio. Conversely, whatever ratio is used, the graph of some exponential is obtained. By properly choosing the ratio we can therefore make the derived graph have any slope we please at J. There must then be some value of b which gives unit slope at J. Denote this value of b by e, then from the definition of a tangent it follows that

$$\operatorname*{Lt}_{x=0}(e^x - 1)/x = 1.$$

The differential coefficients of e^x, a^x, $\log x$ are at once found, and if

$$(-1)^n y_n = 1 - \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} - \dots + (-1)^n \frac{x^n}{\underline{|n}} - e^{-x},$$



we deduce that $dy_n/dx = y_{n-1}$, and hence that $y_1, y_2, \dots y_n$ are all positive if x is positive, so that e^{-x} lies between s_n and s_{n+1} where

$$s_n = 1 - \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} - \dots + (-1)^n \frac{x^n}{\underline{|n}}.$$

Hence as close an approximation as we please can be found for e^{-1}, and therefore also for e. E. J. Nanson.

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THE

MATHEMATICAL GAZETTE.

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THE NORMAL LAW OF ERROR.

1. The investigations of Professor Karl Pearson and his coadjutors on the applications of the theory of errors to statistics have greatly increased the importance and extent of the subject.

It is hoped that the following version of one of the methods of arriving at the so-called normal law of error may be of interest from an elementary point of view: affording, as it does, a simple application of the binomial theorem and some interesting graphical work.

Let each single observation of a quantity, for instance the measurement of a length, be subject to $2n$ errors, each of amount z and each equally likely, in any one case, to be positive or negative.

If $n+r$ errors are positive and $n-r$ are negative, the total, or resultant error will be $2rz$. Now the number of permutations taken all together of $(n+r)$ + signs and $(n-r)$ − signs is $\frac{2n!}{(n+r)!(n-r)!}$, and each permutation represents a mode of occurrence of the set of errors, the chance of that particular permutation occurring being $\frac{1}{2^{2n}}$.

Hence the chance of a total error $2rz=\frac{2n}{n+r!\,n-r!}\,\frac{1}{2^{2n}}$. In other words, the chances of the various total errors

$$2nz,\ 2\overline{n-1}z\ \ldots,\ 2rz\ \ldots,\ -2nz,$$

are given by the successive terms in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^{2n}$.

2. Let us examine the mean value, or expectation, say σ^2, of the square* of the total error.

Multiplying the chance of each total error by the square of its amount, and summing the results we have

$$\sigma^2 = \sum_{r=-n}^{r=n} \frac{2n!}{(n+r)!(n-r)!}\,\frac{1}{2^{2n}}\,4r^2z^2$$

$$= \sum_{r=-n}^{r=n} \frac{2n!}{(n+r)!(n-r)!}\,\frac{1}{2^{2n}}\left[n^2-(n^2-r^2)\right]4z^2$$

$$= \frac{4n^2z^2}{2^{2n}} \sum_{r=-n}^{r=n} \frac{2n!}{(n+r)!\,(n-r)!} - \sum_{r=-n}^{r=n} \frac{2n!}{(n+r-1)!\,(n-r-1)!}\cdot\frac{1}{2^{2n}}\cdot 4z^2$$

$$= \frac{4n^2z^2}{2^{2n}}(1+1)^{2n} - \frac{2n\,.\,2n-1}{2^{2n}}\,4z^2(1+1)^{2n-2}$$

$$= 4n^2z^2 - 2n\,.\,(2n-1)z^2$$

$$= 2nz^2.$$

If then n and z are connected by the equation

$$2nz^2 = \text{constant} = \sigma^2,$$

the "standard deviation" σ will be the same for all values of n.

3. It is interesting to represent the chances of errors of given amount graphically, taking different values of n and z, and keeping σ constant.

Thus we may take $\sigma = 4$ inches,

and the cases $n = 8,\ 32,\ 18,$

$z = 1,\ \tfrac{1}{2},\ \tfrac{2}{3}.$

Taking the first case the maximum ordinate is proportional to $\frac{16!}{8!8!}$ and taking 10 inches to represent this, we have, if $r=0$, say $y_0 = 10$ inches.

If $r=1$, y_1 is proportional to $\frac{16!}{7!9!}$ and therefore $y_1 = \frac{8}{9}y_0 = 8{\cdot}89$, y_1 corresponding to a total error x_1 of 2 inches.

If $r=2$, y_2 is proportional to $\frac{16!}{6!10!}$ and therefore $y_2 = \frac{7}{10}y_1 = 6{\cdot}22$ and so on. Using a slide rate the sucessive ordinates are thus rapidly worked out, and the total error x, and the corresponding chance, y, plotted.

Taking in the same way one of the other cases, we find that a fair curve through the second set of points passes very close to the first set also. In other words, provided σ is constant and n

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*It is natural to ask why the *square* should be taken. Prof. Pearson, *Grammar of Science*, p. 386, cites the analogy of the radius of gyration k: k^2 being the mean square of the distance. Of course the simple mean value or expectation of the total error is zero, for positive and negative errors of equal amounts are equally likely.

is fairly large, the chance of getting an error of a given amount at a single trial is, on the above hypothesis, almost independent of the value of n.

4. Let us now prepare to suppose n to become infinite and z infinitely small in such a manner that $2nz^2$ retains the constant value σ, and r to become infinite in such a manner that $2rz=x$.

Let y denote the relative* chance of a total error $x=2rz$,
and $y+\Delta y$ „ „ „ $x+\Delta x=(2r+2)z$

Thus as before

$$y=\frac{2n'}{(n+r)!(n-r)!}\,\frac{1}{2^{2n}},$$

$$y+\Delta y=\frac{2n!}{(n+r+1)!(n-r-1)!}\,\frac{1}{2^{2n}},$$

whence, by division,

$$1+\frac{\Delta y}{y}=\frac{n-r}{n+r+1} \quad\text{or}\quad \frac{\Delta y}{y}=\frac{-2r-1}{n+r+1}=\frac{-x-z}{nz+\frac{1}{2}x+z},$$

or, since $\Delta x=2z$,

$$\frac{1}{y}\,\frac{\Delta y}{\Delta x}=\frac{-x-z}{2nz^2+xz+2z^2}=\frac{-x-z}{\sigma^2+xz+2z^2}.$$

Now proceed to the limit as above indicated and we have

$$\frac{1}{y}\,\frac{dy}{dx}=-\frac{x}{\sigma^2};$$

or, integrating, $\log y=-\frac{x^2}{2\sigma^2}+C,$

or $y=y_0e^{-\frac{x^2}{2\sigma^2}}.$

Taking $y_0=10$ inches,
$\sigma=4$ inches,

and plotting the curve, we find once more that it passes very closely to the points already found by giving n the finite value 32. If we adjust the value of y_0 so that the whole area of the curve is unity, which should be so, for the area represents the chance of obtaining *some* error between + and − infinity (zero included), we find

$$y_0=\frac{1}{\sigma.\sqrt{2\pi}}.$$

It is now possible to plot, in their proper relative proportions, the curves corresponding to *different* values of σ.

---

* Of course the *absolute* chance of an error being *exactly* x is zero. The chance that the error lies between x and $x+\Delta x$ is $y\Delta x$ nearly.

Abundant illustrations of the application to statistics of the properties of the Normal Curve of Error

$$y=\frac{1}{\sigma\cdot\sqrt{2\pi}}e^{-\frac{x^2}{2\sigma^2}}$$

may be found in Professor Pearson's *Grammar of Science*, and the series of papers in the *Phil. Trans.* already alluded to. Mr. W. F. Sheppard proves many of the properties by elementary methods in a paper in the *Philosophical Transaction*, vol. 192 A, p. 101.

C. S. JACKSON.

THE POWER SERIES FOR $\sin x$, $\cos x$.

THE hopeless complexity, from the elementary didactic point of view, of the hitherto accepted accurate trigonometrical proofs of the power series for $\sin x$, $\cos x$ has been so fully insisted upon in these pages (this volume, pp. 85, 86), that a more elementary method seems urgently called for. The series in question follow at once from the two theorems

$$\sin x \text{ lies between } S_n,\ S_{n+1}, \qquad\qquad (1)$$

$$\cos x \text{ lies between } C_n,\ C_{n+1}, \qquad\qquad (2)$$

where

$$S_n=\frac{x}{\underline{|1}}-\frac{x^3}{\underline{|3}}+\ldots+(-1)^n\frac{x^{2n+1}}{\underline{|2n+1}},$$

$$C_n=1-\frac{x^2}{\underline{|2}}+\ldots+(-1)^n\frac{x^{2n}}{\underline{|2n}}.$$

Now it may be shown by elementary methods that (1) follows from (2) for all real values of x. It may also be similarly shown that if (1) be granted, then $\cos x$ lies between C_{n+1}, C_{n+2} provided $-2\pi \leqslant x \leqslant +2\pi$. Since then it is known that (2) is true when $n=0$ for all real values of x, we at once obtain an inductive proof of (1), (2) for $-2\pi \leqslant x \leqslant +2\pi$.

To show that (1) follows from (2) let there be m angles $\alpha, \beta, \gamma, \ldots$, then

$$\cos\alpha\cos\beta\cos\gamma\ldots=\frac{1}{2^m}\Sigma\cos(\pm\alpha\pm\beta\pm\gamma\pm\ldots)$$

Hence, according as $\cos x \gtrless a_0+\Sigma a_r x^{2r}$,

we have $\cos\alpha\cos\beta\cos\gamma\ldots \gtrless a_0+\Sigma a_r T_r$,

where $T_r=\frac{1}{2^m}\Sigma(\pm\alpha\pm\beta\pm\gamma\pm\ldots)^{2r}$.

Now taking $\alpha, \beta, \gamma, \ldots$ to be $\frac{x}{2}, \frac{x}{2^2}, \ldots, \frac{x}{2^m}$, we have

$$T_r=\frac{2x^{2r}}{2^{m(2r+1)}}t_r,$$

where t_r is the sum of the $2r^{\text{th}}$ powers of the odd numbers from 1 to 2^m-1, so that

$$t_r=\frac{1}{2r+1}\frac{2^{m(2r+1)}}{2}(1+\epsilon),$$

where ϵ can be made as small as we please by taking m large enough. Thus we have

$$T_r=\frac{x^{2r}}{2r+1}(1+\epsilon).$$

Hence, since $\cos\frac{x}{2}\cos\frac{x}{2^2}\ldots\cos\frac{x}{2^m}=\dfrac{\sin x}{2^m\sin\frac{x}{2^m}}$

it follows that $\dfrac{\sin x}{2^m\sin\frac{x}{2^m}}\gtreqless a_0+\Sigma\dfrac{a_r}{2r+1}(1+\epsilon)x^{2r}$,

according as $\cos x\gtreqless a_0+\Sigma a_r x^{2r}$.

Now taking $a_r=\dfrac{(-1)^r}{|\underline{2r}}$, $r=0, 1, 2, \ldots n$

and making m infinite, we see that (1) follows from (2).

To prove the other result which has been stated we require the elementary algebraic inequality that if u lies between U_n, U_{n+1} and v lies between V_n, V_{n+1}, where

$$U_n=u_0-u_1+u_2-\ldots+(-1)^n u_n,$$
$$V_n=v_0-v_1+v_2-\ldots+(-1)^n v_n,$$

then uv lies between W_n, W_{n+1}, where

$$W_n=w_0-w_1+w_2-\ldots+(-1)^n w_n,$$
$$w_r=u_0v_r+u_1v_{r-1}+\ldots+u_rv_0,$$

provided that, for $r>1$, u_r, v_r are positive and $v_r>v_{r+1}$.

Using this result, it follows from (1) by taking

$$u_r=v_r=\frac{x^{2r+1}}{2^{2r+1}|\underline{2r+1}},$$

that $\sin^2\frac{x}{2}$ lies between W_n, W_{n+1}, where

$$w_r=\frac{x^{2r+2}}{2^{2r+2}}\Sigma\frac{1}{|\underline{2p+1}\,|\underline{2q+1}},$$

Σ denoting summation for $p+q=r$, $p=0, 1, 2, \ldots r$, so that

$$w_r=\frac{x^{2r+2}}{2\,|\underline{2r+2}}.$$

Hence, using the formula $\cos x=1-2\sin^2\frac{x}{2}$ it follows from (1) that $\cos x$ lies between C_{n+1}, C_{n+2}, provided $-2\pi\leqslant x\leqslant+2\pi$.

E. J. Nanson.

III. ON CONVERGENCE OF SERIES.

HAVING established the expansion of functions without any reference to tests of convergence, we are able to enunciate the following simple test for the convergence of any series that may arise.

(We confine our attention to a series of which all the terms that we need to consider have the same sign.)

u_n being the n term of a series,

If $\frac{u_n}{u_{n-1}}$ can be expanded in (integral or fractional) powers of $1/n$, none of the terms after n^{-1} affect the convergence.

If $\frac{u_n}{u_{n-1}} \underset{<}{\equiv} 1-\frac{a}{n} \ldots$ where a is greater than 1, the series is convergent.

If $\frac{u_n}{u_{n-1}} \underset{>}{\equiv} 1-\frac{1}{n} \ldots$, the series is divergent.

1. The following theorem is well known: If Σu_i is a series, having $\frac{u_i}{u_{i-1}} \equiv \rho_i$, and Σv_i is another series having $\frac{v_i}{v_{i-1}} \equiv \sigma_i$, then if $\rho_i < \sigma_i$ for all values of i, and Σv is convergent, Σu is convergent; and if $\rho_i > \sigma_i$ and Σv is divergent, Σu is divergent.

For $$\sum_{n+1}^{N} u_i = u_n[\rho_{n+1} + \rho_{n+1}\rho_{n+2} + \ldots + \rho_{n+1}\rho_{n+2} \ldots \rho_N],$$

and $$\sum_{n+1}^{N} v_i = v_n[\sigma_{n+1} + \sigma_{n+1}\sigma_{n+2} + \ldots + \sigma_{n+1}\sigma_{n+2} \ldots \sigma_N].$$

If, therefore, every $\rho <$ the corresponding σ, $\Sigma u < \frac{u_n}{v_n} \Sigma v$;

$\therefore$ if Σv is finite when N is infinite, so is Σu.

That is, Σu is convergent if Σv is.

Similarly, if every $\rho >$ the corresponding σ, and Σv is divergent, so is Σu.

Thus, if we note the values of ρ for known series we can determine the convergence of unknown series.

2. If u_i is any 'derived' function $f'(i)$ which increases or diminishes with increasing i,

$$fi - f(i-1) < \left|\begin{matrix} f'i \\ f'(i-1) \end{matrix}\right|, \text{ and } f(i+1) - fi < \left|\begin{matrix} f'(i+1) \\ f'i \end{matrix}\right|;$$

$$\therefore f'i < \left|\begin{matrix} f(i+1) - fi \\ fi - f(i-1) \end{matrix}\right|; \quad \therefore \sum_{n+1}^{N} f'i \overline{<} \left|\begin{matrix} f(N+1) - f(n+1) \\ fN - fn \end{matrix}\right|.$$

If, therefore, fN is infinite when N is indefinitely increased $\Sigma f'i$ is divergent, but if fN is finite, the series is convergent.

3. *Examples of known series.*

A. If $fx \equiv x^p$, then $f'x \equiv px^{p-1}$;

$$\therefore \sum_{n+1}^{N} i^{p-1} < \frac{1}{p}\left|\begin{matrix}(N+1)^p-(n+1)^p\\ N^p-n^p\end{matrix}\right|.$$

If p is negative, the series is convergent, because N^p is zero; and $\rho_n \equiv \left(1-\frac{1}{n}\right)^{-p+1}$, which $>1-\frac{1-p}{n}$;

$\therefore$ any series for which $\rho_n \leqq 1-\frac{1+a}{n}$ where a is positive, is convergent.

B. If $p=0$, $\frac{N^p}{p}$ becomes indeterminate; the form $\frac{1}{0}(N^0-1)$ suggests the logarithm; but the series in this form is better investigated afresh.

Thus if $fx \equiv \log x$, $f'x \equiv \frac{1}{x}$,

$$\text{for } \log x-\log(x-1) = -\log\left(1-\frac{1}{x}\right) < \left|\begin{matrix}\frac{1}{x-1}\\ \frac{1}{x}\end{matrix}\right|;$$

$$\therefore \sum_{n+1}^{N} \frac{1}{i} < \left|\begin{matrix}\log(N+1)-\log(n+1)\\ \log N-\log n\end{matrix}\right|.$$

$$\therefore \Sigma\frac{1}{i} \text{ is (logarithmically) infinite.}$$

Also, $\rho_n = 1-\frac{1}{n}$.

Hence any series for which $\rho_n \geqq 1-\frac{1}{n}$ is divergent.

We have now to see for what values of a *infinitesimally* greater than 0, $1-\frac{1+a}{n}$ belongs to a convergent series, *e.g.* $a=n^{-\beta}$ or $a=(\log n)^{-\beta}$, where β is positive.

C. For this purpose it is natural to assume fN infinite in some smaller degree. Let fx be $(\log x)^p$.

$$\text{Now } (\log x)^p-(\log y)^p < p(\log x-\log y)\left|\begin{matrix}(\log x)^{p-1}\\ (\log y)^{p-1}\end{matrix}\right|$$

$$< p(x-y)\left|\begin{matrix}\frac{1}{y}\\ \frac{1}{x}\end{matrix}\right|\left|\begin{matrix}(\log x)^{p-1}\\ (\log y)^{p-1}\end{matrix}\right|.$$

Hence, if $(p-1)$ is negative,

$$(\log x)^p-(\log y)^p<p(x-y)\left|\begin{matrix}\frac{1}{y}(\log y)^{p-1}\\ \frac{1}{x}(\log x)^{p-1}\end{matrix}\right|;$$

$$\therefore\ f'x \text{ is in this case } p\,.\,\frac{1}{x}(\log x)^{p-1};$$

$$\therefore\ \sum_{n+1}^{N}\frac{1}{i(\log i)^{p+1}}<\frac{1}{-p}\left|\begin{matrix}(\log\overline{N+1})^{-p}-(\log\overline{n+1})^{-p}\\ (\log N)^{-p}-(\log n)^{-p}\end{matrix}\right|,$$

$\therefore$ the series is convergent if $p>0$, but divergent if p is a negative fraction.

Now, if $p>0$,

$$\rho_n=\frac{n-1}{n}\cdot\left(\frac{\log\overline{n-1}}{\log n}\right)^{p+1}, \text{which} >\left(1-\frac{1}{n}\right)\left(1-\frac{1}{\overline{n-1}\log n}\right)^{p+1},$$

$$\text{which}\qquad >\left(1-\frac{1}{n}\right)\left(1-\frac{p+1}{\overline{n-1}\log n}\right), \text{ that is } 1-\frac{1+\frac{p+1}{\log n}}{n};$$

$\therefore$ any series for which $a\geqq\dfrac{p+1}{\log n}$ where p is positive, is convergent.

D. If $p=0$, $\dfrac{1}{p}\{(\log N)^p-(\log n)^p\}$ becomes indeterminate, but suggests the logarithm, and we find

$$\sum_{n+1}^{N}\frac{1}{i\log i}<\left|\begin{matrix}\log\log(N+1)-\log\log(n+1)\\ \log\log N-\log\log n\end{matrix}\right|;$$

$\therefore$ the series is divergent.

Also

$$\rho_n=\frac{n-1}{n}\cdot\frac{\log\overline{n-1}}{\log n},$$

$$\text{which}\qquad <\left(1-\frac{1}{n}\right)\left(1-\frac{1}{n\log n}\right)<1-\frac{1+\left(1-\frac{1}{n}\right)\frac{1}{\log n}}{n}.$$

$\therefore$ any series for which $a<\left(1-\dfrac{1}{n}\right)\dfrac{1}{\log n}$ is divergent.

Hence, if a is of the order $n^{-\beta}$ where β is positive, the series is divergent.

Hence the rule at the head of this article.

5. This test covers all ordinary cases; but the above investigation continued on similar lines by taking for fx

$$(\log\log x)^p,\ (\log\log\log x)^p, \text{ etc.},$$

leads us to the closer test:

If ρ_n expanded in descending order of magnitude is, or is greater than,

$$1-\frac{1}{n}-\frac{1}{n\log n}-\frac{1}{n\,.\log n\,.\log\log n}\cdots,$$

the series is divergent: if ρ_n is less than this it is convergent.

6. *Examples.* (From Chrystal, Part II., Exercise VIII., pp. 158, 159.)

Ex. 5. $u_n=\dfrac{1}{\sqrt{n^2-n}\{\sqrt{n}-\sqrt{n-1}\}}$,

$$\rho_n=\sqrt{\frac{n^2-3n+2}{n^2-n}}\cdot\frac{\sqrt{n-1}-\sqrt{n-2}}{\sqrt{n}-\sqrt{n-1}}$$

$$=\left(1-\frac{3}{n}+\frac{2}{n^2}\right)^{\frac{1}{2}}\left(1-\frac{1}{n}\right)^{-\frac{1}{2}}\left\{\left(1-\frac{1}{n}\right)^{\frac{1}{2}}-\left(1-\frac{2}{n}\right)^{\frac{1}{2}}\right\}\left\{1-\left(1-\frac{1}{n}\right)^{\frac{1}{2}}\right\}^{-1}$$

$$=\left(1-\frac{3}{2n}\cdots\right)\left(1+\frac{1}{2n}\cdots\right)$$

$$\times\left\{1-\frac{1}{2n}-\frac{1}{8n^2}-1+\frac{1}{n}+\frac{1}{2n^2}\cdots\right\}\left\{\frac{1}{2n}+\frac{1}{8n^2}\right\}^{-1}$$

$$=\left(1-\frac{1}{n}\cdots\right)\left\{1+\frac{3}{4n}\cdots\right\}\left\{1-\frac{1}{4n}\cdots\right\}$$

$$=1-\frac{1}{2n}\cdots;\quad\therefore\ \text{divergent.}$$

Ex. 12. $u_n=\dfrac{1}{n^a}\sum_1^n\dfrac{1}{i^a}$;

$\therefore\ \rho_n=\left(\dfrac{n-1}{n}\right)^a\left\{1+\dfrac{1}{n^aS_{n-1}}\right\}$, where S_{n-1} is written for $\sum_1^{n-1}\dfrac{1}{i^a}$

$$=\left(1-\frac{a}{n}\cdots\right)\left(1+\frac{1}{n^aS_{n-1}}\right).\quad\text{This} > 1-\frac{a}{n};$$

$\therefore$ if $a \leqq 1$, the series is divergent;

but if $a > 1$, we have in the article above,

$$\underset{n-1}{S} < 1+\frac{1}{a-1}\left|\begin{matrix}2^{-a+1}-n^{-a+1}\\1^{-a+1}-(n-1)^{-a+1}\end{matrix}\right|;$$

$$\therefore\ n^a S_{n-1} < \text{const. } n^a - \frac{1}{a-1}\left| n\left(1-\frac{1}{n}\right)^{-a+1} \right|^{n};$$

$\therefore\ \dfrac{1}{n^a S_{n-1}} = \dfrac{\text{const.}}{n^a} + \ldots$, which does not affect the convergence.

$\therefore\ \rho_n = 1 - \dfrac{a}{n} \ldots$, and the series is convergent.

7. If $f(x_i) - f(x_{i-1}) < (x_i - x_{i-1}) \left| \begin{matrix} f'x_i \\ f'x_{i-1} \end{matrix} \right|$,

Then $$\sum_{n+1}^{N} (x_i - x_{i-1}) f'x_i > fx_N - fx_n,$$

and $$\sum_{n+1}^{N} (x_{i+1} - x_i) f'x_i < fx_{N+1} - fx_{n+1}.$$

Cauchy's Condensation Test is the particular case of this when $x_i \equiv k^i$ and $k > 1$.

W. N. ROSEVEARE.

REVIEWS.

Vorlesungen über Technische Mechanik. By Dr. A. FÖPPL. Vol. i. **Einführung in der Mechanik.** 3rd ed. pp. 424. 10 m. Vol. iii. **Festigkeitslehre.** 3rd ed. pp. 434. 12 m. (Teubner, 1905.)

In the first volume of this work Professor Föppl's object is to introduce a student who is familiar with the experimental facts of mechanics, to the mathematical treatment of the subject. A working knowledge of the *meaning* of such symbols as $\dfrac{ds}{dt}$ and $\int Pd$[illegible] is assumed, though little analytical skill is demanded of the reader.

The feature which distinguishes Professor Föppl's treatise from most English works on the same subject is the use of vectorial algebra and not merely vector addition. Both in these volumes and in his Introduction to Maxwell's Theory of Electricity, Professor Föppl constantly employs a vectorial notation and vectorial methods. He adopts the convention that, for scalar multiplication $i^2 = j^2 = k^2 = +1$.

Vector multiplication has at present received little attention from teachers of elementary mathematics in this country. This is probably due in part to the unfortunate controversy as to notation. [See Professor Henrici's communication to the British Association. *B.A. Report*, 1903, p. 55.]

But the use of almost any vector notation is of value as an aid to the memory and as directing attention to a physical quantity in itself instead of merely to its components along the axes.

An analogy is furnished by determinants. Who does not find it easier to recollect the result of eliminating x, y, and z from three equations (1), $a_1x + b_1y + c_1z = 0$, (2), and (3), in the form of a determinant?

To illustrate in a similar manner the use of vector multiplication; suppose a system of forces $R_1 \ldots R_n$ acting at points in space $r_1 \ldots r_n$.

The sum of the vector products $\Sigma[rR]$ gives the couple, the vector $\Sigma(R)$ the force acting at O, to which the system is reducible.

If then
$$R_l = X_l i + Y_l j + Z_l k, \qquad l = 1 \ldots n,$$
$$r_l = x_l i + y_l j + z_l k,$$
since for vector multiplication
$$ij = k, \; jk = i, \; ki = j,$$
and
$$i^2 = j^2 = k^2 = 0,$$
we have $\Sigma[rR] = i\Sigma(Zy - Yz) + j\Sigma(Xz - Zx) + k\Sigma(Yx - Xy)$
and $\Sigma(Zy - Yz)$ is the component couple, in a right-handed system of axes, about the axis of x.

Again, suppose a system of axes moving about a fixed point with angular velocity $\Omega = i\theta_1 + j\theta_2 + k\theta_3$.

Suppose a vector quantity referred to the moving axes, as it would be by an observer unaware of the movement of the axes.

The point which is the geometrical representative so to speak of the vector quantity is supposed, that is to say, to be carried with the moving axes, and also to move relatively to them.

Now if $a = i\xi + j\eta + k\zeta$ is the vector and $q = \frac{d}{dt} + i\theta_1 + j\theta_2 + k\theta_3$ an operator, the vector product
$$\begin{aligned}[q\alpha] = i&(\dot{\xi} - \eta\theta_3 + \zeta\theta_2) \\ +j&(\dot{\eta} - \zeta\theta_1 + \xi\theta_3) \\ +k&(\dot{\zeta} - \xi\theta_2 + \eta\theta_1)\end{aligned}$$
gives the rates of change of a, resolved into components along a right-handed system of axes fixed in space, and coinciding with the original position of the moving axes.

Putting the matter on the lowest ground these expressions appear to furnish a useful aid to the memory.

Professor Föppl's first volume contains many interesting applications—and he is always careful to discuss points of fundamental principle at adequate length.

The contents of the third volume on strength of materials and elasticity may be indicated by the titles of the chapters, which are as follows: General Investigation of Stress Components, Elastic Deformation, Bending of Straight Rods, Energy of Elastic Deformation, Curved Beams, Elastic Supports, "Built in" Plates, Resistance to Internal or External Pressure, Torsion, Fracture, Elements of the Mathematical Theory of Elasticity.

The mode of treatment adopted, making free use of the calculus, is more advanced than that of most English elementary text-books, though the book is not a mathematical treatise on the theory of elasticity.

To those desirous of going a little beyond an elementary discussion of Hooke's Law and bending moments the book should prove very interesting.

Reference may be made especially to the discussion of a beam subject to a non-axial thrust, and to the investigation by Castigliano's theorems of the 'continuous girder' or elastic beam resting on more than two supports. (See *Gazette*, vol. iii., p. 28.)

It may be mentioned that a French translation by M. Hahn, from the first edition of Professor Föppl's third volume, was published in 1901 by M. M. Gauthier-Villars. C. S. JACKSON.

Elementary Dynamics. By W. M. BAKER, M.A., Head Master of the Military and Civil Department of Cheltenham College. Second edition. (London: G. Bell and Sons, 1905. pp. 318. 4s. 6d.)

This is not an easy book to review. On the one hand praise is due to concise style and systematic treatment, and to the number and variety of the examples and their classification and arrangement. One is struck too by the way in which the difficulties of the subject are evaded.

But inasmuch as the subject does present certain difficulties and as the type of mind which never finds a difficulty in any subject has its disadvantages, ought this feature to be praised or blamed?

Mass is "quantity of matter" (p. 31); all motion is relative (p. 111), but force is measured by the product of mass and acceleration (p. 34) and yet is absolute (p. 35); variable velocity is measured "at any instant by the distance which would be passed over in unit of time if the point moved during that unit of time with the same velocity as at the instant under consideration." The rotatory definition last quoted occurs at p. 1, though it is true that velocity at a time t is obtained at p. 21 as the limit of $\frac{\Delta s}{\Delta t}$.

That the work is silent as to the history of dynamics and as to the practical applications of the subject must not be imputed to the author as a fault, for the book is written in view of certain examinations and it is not fair to blame an author who is subject to the conditions of a system for the consequences of that system. One is, however, surprised to read at page 123 "Angular velocity is measured in radians." C. S. JACKSON.

Oblique and Isometric Projection. By J. WATSON. (Edward Arnold. 3s. 6d.)

The principles of these projections are simply explained and their application to draughtsmen's work well illustrated. The diagrams are clear and of good size. We are afraid that mathematical teachers as a rule do not trouble themselves about such matters. This is to be regretted as they thus miss many a good exercise and illustration which they might use with effect when going through a school course of Theoretical Solid. A little of the 'fusing' discussed at one of the General Meetings would be good for both Practical and Theoretical work. E. M. LANGLEY.

An Introduction to Projective Geometry and its Applications, an Analytic and Synthetic Treatment. By ARNOLD EMCH, Ph.D. (New York: Wiley & Sons. London: Chapman & Hall. 1905. pp. 267.)

This seems to us a singularly ineffective book. It must prove disappointing to a reader who wishes to gain from it some definite ideas of projective geometry; and it only differs from any ordinary book on geometry in being more than usually meagre and inaccurate. Projective geometry is the geometry which is founded on projection, and, as generally understood, makes no metrical assumptions, but regards all figures as being immovably fixed in space. It is the most fundamental of all geometries, being based on the fewest axioms. Even if metrical ideas were permitted we should still regard projective geometry in its beginnings as pure geometry. Dr. Emch, however, is of a different opinion; he makes no enquiry as to the foundations and limitations of the science, and defines projective ranges quite apart from projection. His work, as he says, has a utilitarian purpose, although this is not very apparent in the greater part of the book. But, being utilitarian, why should the terms descriptive and graphical be rejected, and a term be chosen which is specially used in a sense opposed to these?

The book contains five chapters devoted respectively to General Considerations, Collineation, Conics, Cubics, and Mechanics. Some of the subsections are involution, perspective, central projection, orthographic projection, polar involution, properties of the centre, diameters, asymptotes, and foci of conics, and the theorems of Pascal and Brianchon. The last chapter includes the geometry of stresses, and a description of a number of linkages.

We can only refer to some of the weak points in the exposition. The questions of the independence and consistency of a given system of linear equations are generally ignored. On p. 132 it is indeed shown that a set of six linear equations are not equivalent to more than five; but it is there stated that they have a unique solution for five unknowns, although the six equations have been formed from four only. The theorem professedly proved is of course false. On p. 23 the enumeration of independent constants is incorrect. The description on pp. 27, 28 of the complete quadrilateral is actually that of the complete quadrangle; and the remarkable theorem is given:—"In every complete quadrilateral a pair of sides always forms a harmonic pencil with the two concurrent diagonals." It would be difficult to cram more mistakes into a statement of equal brevity. Six lines lower down the terms quadrilateral and diagonal are used in the proper sense, *i.e.*, in a sense quite contrary to that in which they have been defined. The terms quadrilateral, quadrangle, and quadruple are elsewhere used indiscriminately. As regards the polar involution of the circle we read (p. 43):—"For every ray through P a pair of poles and a pair of polars are obtained which are harmonic to A and B, and to a and b, respectively. In this manner an *involution of coincident poles and polars* arises." The italics are those of the book. This is the sole explanation of the meaning of polar involution. We fear it will leave the reader, as it leaves us, unenlightened. According to the explanation given on p. 68, the Principle of Duality means nothing more than that a curve may be regarded as a locus of points or as an envelope of lines. Besides the use of quadrilateral for quadrangle, coincident is used for incident (p. 127), and bisection for harmonic section (pp. 89, 142). Definitions are as a rule omitted; and the whole treatment is entirely informal and vague.

In the preface it is stated that Enriques "in his book," *Projektive Geometrie*, "lets the fundamental elements of the first order be generated by motion!" This is misleading, if, as we presume, forms are meant by elements. Enriques only uses motion by way of illustration. His geometry is founded on carefully formulated axioms, in which the idea of motion has no part. F. S. M.

Geometrical Conics. By CAUNT AND JESSOP. (London: Edward Arnold. Price 2s. 6d.)

This is a small book of 80 pages. The authors define the conic as the section of a right circular cone, and from this definition they deduce the focus and directrix property, the bifocal property and the construction for describing the curve by mechanical methods. A short account is given of the polar properties of the circle, on which, by the method of projection, the authors base the theory of pole and polar and of conjugate diameters in the conic. The book deserves commendation for giving prominence to the idea of the continuity of the different species of the conic. We think, however, that the explanation of the method of projection is imperfect. In Chapter I. the conic is defined as the section of a *right* circular cone, but in Chapter III. the section of *any* cone on a circular base is called a conic; and it is not proved that every conic, defined by the focus and directrix property, can be projected into a circle. R. H. PINKERTON.

The First Book of Euclid's Elements, with a Commentary based principally on that of Proclus Diadochus. By W. B. FRANKLAND. Pp. xvi, 139. 1905. (Cam. Univ. Press.)

Mr. Frankland has followed up his excellent *Story of Euclid* by an "anthology of the best commentary on any part of the Elements ever composed." We observe with regret from the preface that untoward circumstances of a private nature have prevented the earlier appearance of the book before us. But such is its quality and that of its predecessor that it is evident that Mr. Frankland possesses a large measure of the equipment required in the writer on such a topic as the parallel postulate. We therefore welcome the hint given us in the first chapter, that some day there will appear from the skilful and learned pen of the author an historical and critical study of that problem. We could wish we had space for many quotations from this witty and charming volume. Here is one which would have gladdened the heart of the author of *Euclid and his Modern Rivals*, and of *Alice in Wonderland*: "Easy introductions to Euclidean dogmatics do not expose clearly and precisely the principles at their bases—they need to be exposed. Such plebeian roads . . . were beginning to appear, like much else reckoned modern, in the *Commentary of Proclus*." "Parallelism is similarity of position. This is good illustration, but bad analysis." We must content ourselves with saying that these "innocent studies" form an eminently valuable addition to the history of Geometry, and we may add, an attractive volume.

Euclid's Parallel Postulate. Its Nature, Validity, and Place in Geometrical Systems. By J. W. WITHERS, Ph.D. Pp. vii, 192. 1905. (Kegan, Paul.)

This book is the thesis presented to the Philosophical Faculty of Yale University for the degree of Ph.D. by the author. He gives an account of the history of the postulate up to the time of the discovery and development of non-Euclidean systems, and dwells upon the significance of the latter. He then embarks on the philosophical side of his theme. First he discusses the psychology of the parallel postulate and its kindred conceptions, showing that its validity is not to be settled by any empirical investigations. Next he deals with the number and variety of possible space geometries. Finally he shows the inferences that follow as to the nature of space when the validity of the parallel postulate is denied. On the whole we may consider the book as a compilation in which the problem and its history are presented accurately and attractively. The author seems to have read everything (in English) that ought to have been read for the purpose, but we do not rise from the perusal of his book with the feeling that anything has been added to what has been said many times before. The book, however, has a certain value, but we are sorry to add that its value is considerably impaired by a large number of errors. Thus we have—"Eudemas" for "Eudemus" (p. 2) and "Germinus" for "Geminus" (p. 5). On p. 6 we are told that the most critical efforts of modern times have not destroyed, but have strengthened Euclid's claims to rigour, an opinion which would find no support from Mr. Bertrand Russell. On p. 7 "Autolycus" becomes "Antolycus," and Sir Henry Savile receives an extra "l" in his surname. The meaning of the Latin quotation is destroyed by "elucendis" for "eluendis" and by an omission. Playfair's axiom (p. 11) is credited to "Ludlum" instead of to "Ludlam." On p. 13 we have La Place, and the writer speaks of T. Perronet Thompson, "of Cambridge," brilliantly stating the insufficiency of twenty-one different attempts to dispose of the parallel postulate. The number should be 30 and not 21, and Thompson could hardly be described as "of Cambridge." It is true that he was at Queen's at the mature age of fifteen, but he went into the Navy at twenty, and is not likely to have done much that was brilliant within that period. After serving in the Navy for three years he exchanged into the Army, and eventually attained the rank of General. His connection with Cambridge can have been but slight, for his work as joint-editor of the *Westminster Review*, and the claims of Parliament (he was M.P. for Hull and for Bradford) must have kept him for the most part in London. On p. 14 we have "geomatriae" and "Legendri." The *Encyclopaedia* reference on p. 19 should be p. 666. On p. 38 (η, θ) should be (r, θ). On p. 43 "ejusdem" is wrongly spelled, and on the next page we have "spazzii" for "spazii" and "constante" for "costante." "Motion" for "notion" on p. 143 spoils the sense of the sentence. In the Bibliography at the end we think we are correct in stating that not a single French accent is rightly placed. Under the name Beltrami we have "spazii" rightly spelled this time, but "constanta" again, and "interpretazioni" for "interpretazione." Under *Bolyai* there is "subliminis" for "sublimioris" and "evidentique" for "evidentiaque." Under *Bonola* we have "Matematische" for "Matematiche" and "Fondamenta" for "Fondamenti." Under *Calinon* there is "Matematique" for "Mathématiques." The firm of Macmillan is throughout given as MacMillan. *Couturat* is omitted from the list. The reference to *Mind* under the title *Helmholtz* should be Vol. II. and not III. Under *Klein* we have "Reimann,' and under *Pasch* "neure" for "neue," "Euclidis" for "Euclides" and "philosphe" for "philosophie"—but why extend the tale? The author either knows nothing of the languages from which he quotes or has been amazingly indifferent to the necessity of correcting his proofs.

Easy Mathematics, chiefly Arithmetic; being a collection of Hints to Teachers, Parents, Self-taught Students, and Adults, and containing Most Things in Elementary Mathematics Useful to be Known.

By Sir Oliver Lodge. Pp. xv, 436. 1905. (Macmillan.)

There is a quaint eighteenth-century flavour about the sub-title of an interesting volume, which owes its existence to the author's conviction that mathematics may be taught very much better than is the case in most of our schools. Accordingly he has tried to make the subject lively and interesting. "Dulness and bad teaching are synonymous terms." Teaching that is not characterised by these qualities is actually harmful. "A few children are born mentally deficient, but a number are gradually made so by the efforts made to train their growing faculties." These be bitter words, and we devoutly hope the indictment is undeserved, at any rate with regard to a large number of our secondary schools. Sir Oliver's wide experience and the prominent position he has won for himself in the world of science, together with the opportunities for observation he enjoys in the important position he now holds, entitle his views to be carefully studied and with respectful attention, and most teachers will read with pleasure a book written in such a breezy style, thoroughly sane from cover to cover; while few will find no hint in its pages by which their attitude towards the unformed mind as teachers of the subject is not capable of improvement. Some of the points made are quite excellent. Among the examples on the unitary method are some which most certainly will find out for a teacher if his boys are applying their method in a thinking way. "If two peacocks can awaken one man, how many can awaken six? If a camel can stand a load of 5 cwt. for six hours, how long can he stand a load of ten tons?"

Cours de Géométrie Analytique de deux Dimensions. Des Sections Coniques. By H. MANDART. Pp. 574. 1905. (Wesmael, Namur.)

We are glad to draw attention to this volume on Analytical Conics for two reasons. In the first place we do not come into close enough contact with the really excellent work which is being done in the Belgian secondary schools, technical schools, and colleges, and we get little opportunity of seeing with what apparatus of text-books are obtained results which are making Belgium a force to be reckoned with in the industrial and scientifical world. And in the next place for the sake of the book itself, inasmuch as it initiates a departure from the usual traditionary treatment of the subject. The books on Conics for the Belgian secondary schools which we have seen have seemed to differ in some respects from the more advanced works intended for the consumption of children of a larger growth. In the former case there is no attempt to impart general views; the treatment differs but little from that to which we are accustomed, and the geometrical point of view reigns supreme. The conics are discussed each in turn; rarely does the general equation appear. But directly the student makes his way to the Universities he has to begin the subject over again. He meets with the ternary quadratic form; projective properties are utilised for the relations between the partial derivatives of the form, and the metrical properties are treated with the aid of the cyclic points. For some time past M. Mandart has been dissatisfied with this way of dealing with the elements of the theory of curves of the second order, and the volume under notice is the outward and visible sign of the methods of teaching which he has been trying for some years past. Let us say in a nutshell that the whole aim and object of the book is to keep before the student the rôle of invariants, while at the same time using nothing but the elements of invariantology. Whether the method will prove satisfactory from the teaching point of view or not we will not attempt to hazard a conjecture. It might be tested, first with picked boys, and then with the rank and file if the results seem promising. The book forms a complete treatise. The first two parts deal with the point, line, and circle, and with lines of the second degree. The third takes up the general theory of conics, and the fourth is devoted to homogeneous trilinear coordinates. General principles are always kept in sight, and nothing is slurred or passed over without close examination. Plenty of examples are worked, and plenty set for the student as exercises. The whole subject is treated in a manner which reveals the hand of a teacher with a mastery of his work, so complete and homogeneous is the book, and so lucid is the exposition. We would like to hear if any teacher finds it possible to make even a partial attempt to teach Analytical Conics on these lines, feeling sure that the experiment is worth a trial.

Mathematical Recreations and Essays. By W. W. ROUSE BALL. Pp. 388. 4to Edition. 1905.

It is about three hundred years since Claude Gaspar Bachet, Sieur de Meziriac, wrote in his dedication to Monsieur le Comte de Tournon of his *Problèmes Plaisants et Delectables*: "Je vous offre des jeux, mais qui sont, à mon avis, dignes de votre bel esprit, et capables de lui fournir quelquefois un agréable divertissement." One hundred years later Leibnitz, in a letter to Montfort, expressed the wish "qu'on eût un cours entier des jeux, traités mathématiquement." In 1694 appeared Ozanam's *Recréations Mathématiques et Physiques*, which was a compilation from various sources, Ozanam's own share being of but indifferent value. Still, his book was a success and went through many editions, the last being in one volume in 1840. In 1892 Mr. W. W. Rouse Ball published a volume entitled *Mathematical Recreations and Problems of Past and Present Times* (Messrs. Macmillan). A fourth edition has now made its appearance with the 240 pages of the original increased to almost 400. There are three new chapters, dealing respectively with the History of the Mathematical Tripos at Cambridge, Mersenne's Numbers, Cryptography and Ciphers. Part II., which was headed "Mathematical Problems and Speculations," is now altered to "Miscellaneous Essays and Problems." The problem of Mersenne's numbers is to find the values of p which make $2^p - 1$ a prime—*i.e.* it is a particular case of finding the factors of $a^n - 1$. The peculiar interest attaching to Mersenne's researches is that to this day no one can say how he or his contemporaries arrived at their results. Copious references in this chapter direct the student where to look for information on the various cases, and it is with mournful resignation that we learn that no machines have yet been devised for the detection of primes among numbers so large as those given by Mersenne. The chapter on Cryptography and Ciphers is full of information. Here we have the historical ciphers of Julius Caesar, Augustus, Bacon, Charles I., Pepys, and—sinister conjunction of names—those of De Rohan and Marie Antoinette, though the De Rohan is not the great Cardinal, but one of humbler rank—a mere Chevalier, who probably lost his life through being unable to read the cipher message sent him by his friends. As there may be occasions in the lives of the least adventurous among us when it may be useful and advisable to baffle the curiosity of an occasional village postmistress, there will be times when this chapter will be eagerly read.

The old chapter on the "Constitution of Matter" has been reconstructed, as, indeed, was inevitable in the light of recent researches. The sections added deal with the hypothesis of an elastic solid aether and Sir William Thomson's labile aether. The dynamical theories now treated are the hypothesis of the vortex ring, of the vortex sponge, of the aether squirt, of the electron, and of the bubble. There is also a short account of the latest speculations due to investigations on radio-activity. Some account might have been given of Le Bon's speculations, a convenient summary of whose suggestive book has been given recently in the *Athenæum*. Roughly speaking, his theory may be summed up as follows: All matter is slowly dissociating; electricity is the intermediate state through which matter passes on its passage to the aether from whence it came. But to return to Mr. Rouse Ball. We might naturally expect from the author of *The History of the Study of Mathematics at Cambridge* a lucid and systematic account of the Tripos, and we are not disappointed. We see how the word has changed its meaning as few words have, "from a thing of wood to a man, from a man to a speech, from a speech to sets of verses, from verses to a sheet of coarse foolscap paper, from a paper to a list of names, and from a list of names to a system of examination." With the author, we regret to note the disappearance during the last ten years of the famous Tripos verses, which in their time caused a good deal of harmless and innocent amusement. "Mr. Tripos" had been an unchartered libertine for three hundred and twenty years, off and on; he was allowed to say anything he liked in these verses "so long as it was not dull and was scandalous"; and, to quote the exhortation of the University officials, he always remembered of recent years while exercising his privilege of humour to be modest withal. As well might we call for the abolition of the Westminster Play! We must add that a considerable amount of new matter has been added in this edition, much of it of an elementary character and such as to be easily understanded of the vulgar.

W. J. G.

On the Traversing of Geometrical Figures. By J. Cook Wilson. 1905. (Clarendon Press.)

To men who read mathematics occasionally for amusement, but whose serious work has usually lain in other directions, this book will almost certainly be of interest. It needs no close or concentrated attention, and deals in a simple and convincing manner with the discussion of a mathematical puzzle which may easily have already occurred to the reader as one deserving of solution. Under what circumstances is it possible to trace out a given geometrical figure without removing the pen from the paper? For the solution arrived at, which is very simple, and kindred topics, the reader must be referred to the book itself. Part III. of the treatise is devoted to the application of the Principle of Duality, and the "Addendum," published subsequently to the treatise, deals with the same matters. Here the author is more difficult to follow, and indications are not wanting that he is not himself one of the elect to whom the inner mysteries of modern Pure Mathematics have been laid bare.

W. H. Young.

MATHEMATICAL NOTES.

177. [K. 22. c.] *Projection of Diagrams.*

In an article on "Projection of Mathematical Diagrams" in the *Penny Cyclopædia*, De Morgan remarks:

"The diagrams by which mathematical students (and even writers) represent their solid figures are generally so imperfect that it may be worth while to explain how in all cases of sufficient importance a good drawing may be made with very little trouble. . . ."

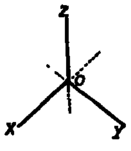


FIG. 1.

In this Note De Morgan is followed closely.

The projection is supposed to be ORTHOGRAPHIC, in which the eye is at an infinite distance, and all parallels project into parallels, etc.

Let OX, OY, OZ (Fig. 1) be the intended projection of the three axes of coordinates $O'X'$, $O'Y'$, $O'Z'$, the dark lines being supposed to belong to that quarter of space in which lies a line drawn from the eye to the origin. Each of the angles YOZ, ZOX, XOY is then greater than a right angle. Let these angles be denoted by X, Y, Z, and let the projections of a line of given length l in the directions of $O'X'$, $O'Y'$, $O'Z'$, be x, y, z, then

$$x:y:z=\sqrt{-\sin 2X}:\sqrt{-\sin 2Y}:\sqrt{-\sin 2Z}.$$

For the proof De Morgan refers to the *Cambridge Mathematical Journal*, No. 8, p. 92.

But we may proceed as follows:

Let A, B, C be the points in which $O'X'$, $O'Y'$, $O'Z'$ meet the plane of projection. Draw $O'D$, $O'E$, $O'F$ perp. to BC, CA, AB and join AD, BE, CF.

$$AB^2=AO'^2+O'B^2=AO'^2+O'D^2+BD^2=AD^2+DB^2,$$

$$\therefore\ AD \text{ is perp. to } BC;$$

$$\therefore\ O \text{ lies on } AD.$$

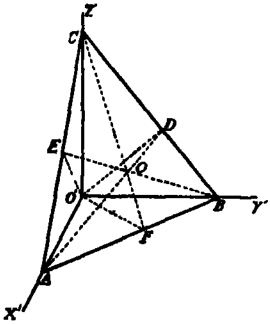


FIG. 2.

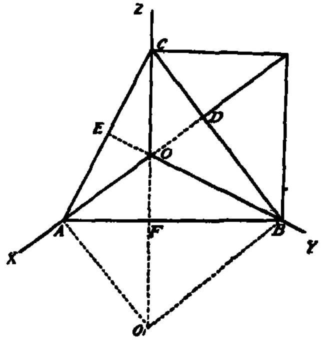


FIG. 3.

Similarly O lies on BE and CF, *i.e.* O is the orthocentre of ABC.

$$\therefore\ \angle BOC=180^\circ-A,$$

$$\text{and } \therefore\ AO'^2=AE.AC=AO.AD.$$

(In Fig. 3 the triangle $AO'B$ is folded into the plane ABC, which is taken as that of the paper.)

Then
$$\frac{AO^2}{AO'^2}=\frac{AO^2}{AF.AB}=\frac{AO^2}{AO.AD}=\frac{AO}{AD}=\frac{AO.BC}{2\Delta}$$
$$=\frac{2R\cos A.2R\sin A}{2\Delta}$$
$$=\frac{R\sin 2A}{\Delta}$$
$$=-\frac{R}{\Delta}\sin 2X;$$
$$\therefore\ x=l\sqrt{-\frac{R}{\Delta}\sin 2X}.$$

Similarly for y and z.

For convenience take $l\sqrt{\frac{R}{\Delta}}$ as unity, then the following table gives the length of the projections for the angles entered.

91°	·187	106°	·728	121°	·940
92	·264	107	·748	122	·948
93	·323	108	·767	123	·956
94	·373	109	·785	124	·963
95	·417	110	·802	125	·969
96	·456	111	818	126	·975
97	·492	112	·834	127	·980
98	·525	113	·848	128	·985
99	·556	114	·862	129	·989
100	·585	115	·875	130	·992
101	·612	116	·888	131	995
102	·638	117	·900	132	·997
103	·662	118	·911	133	·999
104	·685	119	·921	134	1·000
105	·707	120	·931	135	1·000

Since $\sin 2(270^\circ-A)=\sin(540^\circ-2A)=\sin 2A$,
the length corresponding to an angle greater than 135° may be found by taking the reading for its defect from 270°; thus the length corresponding to 160° is ·802, the same as for 110°.

The use of the table is as follows:

Suppose the angles X, Y, Z to be 123°, 105°, 132°, as in Fig. 1. Opposite the angles put down the numbers belonging to them in the table, and opposite to each number the corresponding coordinate

X	123°	·956	x
Y	105°	·707	y
Z	132°	·997	z

These numbers show the proportions which the projections of equal lines bear to one another on the three axes. Thus

$$\frac{\text{Projection of 1 ft.} \parallel \text{to } O'X'}{\text{Projection of 1 ft.} \parallel \text{to } O'Y'}=\frac{\cdot 956}{\cdot 707}.$$

The isometrical perspective of Prof. Farish is the simplest case of this, namely, that in which the angles are each 120°.

See Sopwith's "Isometrical Drawing" for an elaborate account of this and its various applications.

E. M. LANGLEY.

178. [A. 1. a.] *Approximation to the r^{th} root of a number.*

In a review published in the May number of this *Gazette*, Mr. C. S. Jackson has suggested methods of arriving at a certain approximation to the r^{th} root of a number N. The following method of dealing with this problem is very elementary, in that it only assumes the Binomial Theorem *for a positive integral index* and it is closely allied to the ordinary arithmetical method of extracting roots.

Let a be the first approximation to the root, and let

$$\sqrt[r]{N}=a+x.$$

$$\therefore\ N=a^r+r\,.\,a^{r-1}x+\frac{r(r-1)}{2}a^{r-2}x^2+\dots,$$

$$\text{i.e. } x=a\,.\,\frac{N-a^r}{r\,.\,a^r}\Big/\left(1+\frac{r-1}{2}\frac{x}{a}+\frac{(r-1)(r-2)}{2\,.\,3}\frac{x^2}{a^2}+\dots\right).$$

The first approximation for x is 0.

The second is $x_2=a\,.\,\frac{N-a^r}{r\,.\,a^r}$. This is the approximation which is generally used in seeking the next digit in arithmetical work.

The third approximation is

$$x_3=x_2\Big/\left(1+\frac{r-1}{2}\frac{x_2}{a}\right)=\frac{2a(N-a^r)}{(r+1)a^r+(r-1)N}.$$

The corresponding approximation to $\sqrt[r]{N}$ is $a+x_3$, which is

$$a\,.\,\frac{(r-1)a^r+(r+1)N}{(r+1)a^r+(r-1)N}.$$

This solution can therefore be obtained without any preliminary assumption as to its form.

It is of some interest to notice that this work can be set out in the way which is generally used for square and cube root, thus:

$$\left[\text{N.B.}-x_2\equiv\frac{N-a^r}{ra^{r-1}}\right]$$

$$\begin{array}{r|l} N & a+x_2-\frac{(r-1)\,.\,x_2^2}{2a}\dots \\ a^r & \\ \hline r\,.\,a^{r-1}+\frac{r(r-1)}{2}a^{r-2}x_2+\dots & (N-a^r) \\ & \underline{(N-a^r)+r(r-1)x_2^2a^{r-2}/2+\dots} \\ r\,.\,a^{r-1}+\dots\dots\dots\dots & -r(r-1)x_2^2a^{r-2}/2-\dots \\ & \underline{-r(r-1)x_2^2a^{r-2}/2-\dots.} \end{array}$$

When the higher powers are neglected this is equivalent to

$$\sqrt[r]{N}\doteqdot a+x_2-\frac{r-1}{2}\frac{x_2^2}{a}$$

$$\doteqdot a+x_2\Big/\left(1+\frac{r-1}{2}\frac{x_2}{a}\right),$$

the same result as before.

F. J. W. WHIPPLE.

179. [A. 3. k.] *Cubic Equations.*

Attention may be called to a Note by Mr. D. Biddle on a direct method of solving (to any required degree of approximation) cubic equations when irreducible by Cardan's method. It is to be found in the *Educational Times* for July 1st, 1905. The process is "neither laborious nor prolonged, and though expedited by the use of logarithms and reciprocals, is in reality independent of all tables."

180. [K. 2. a.] *Note on Simson's Line.*

A, B, C, D are points on a circle, centre O, radius R; K, k orthocentres of the triangles ABC, DBC.

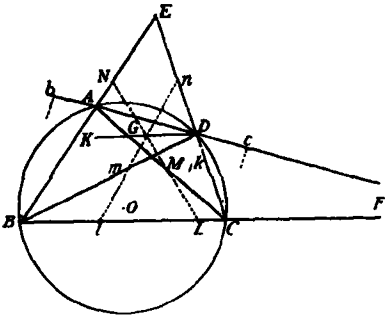
Then AK, Dk are parallel and each equal to twice the perpendicular from O on BC, hence $AKkD$ is a parallelogram, and its diagonals bisect each other in G.

The rectangular hyperbola through A, B, C, D must also pass through K and k; so that G is its centre, and the nine-point circles of the triangles ABC, DBC, ABD, ACD intersect in G.

If LMN, lmn be the Simson lines of D and A with respect to the triangles ABC, DBC respectively, we may assume as known that LMN bisects DK, *i.e.* it passes through G.

Similarly the pedal lines of A, B, C with respect to the triangles DBC, ACD, ABD also pass through G.

Let BA, CD meet in E; and BC, AD meet in F.



Now $DNA = AnD = 90°$, $\therefore$ $ANnD$ is cyclic, and Nn, AD are antiparallel with respect to EAB, EDC. Hence Nn is parallel to BC, and in the same way Mm is parallel to BC.

The perpendicular from G on BC bisects Ll; $\therefore$ $GL = Gl$, $GM = Gm$, and $GN = Gn$.

Thus the four Simson lines not only co-intersect in G, but are all of the same length, and have their segments measured from G to the sides of the respective triangles all equal to one another.

Let Bb, Cc be perpendiculars to AD; then

$$Gb = Gc = Gl = GL.$$

Thus G is the circum-centre of the triangle lbc, which is similar to ABC (for $\angle lbc = \angle lBA$ and $\angle lcb = \angle BCA$).

Hence G is the centroid of masses $\sin 2A$, $\sin 2B$, $\sin 2C$ at l, b, c respectively.

But $$\angle Abm = \angle ABD = \angle ACD = \angle Acn.$$

$$\therefore\ bm \text{ is parallel to } cn.$$

Thus G is also the centroid of masses $\sin 2A$, $\sin 2B$, $\sin 2C$ at l, m, n respectively, or (as the respective distances from G are equal) of those masses at L, M, N. Similar results hold for the other similar lines.

Lastly, let us take the proposition for which Mr. R. F. Davis (*Math. Gazette*, No. 48, p. 116) desires a geometrical meaning.

Suppose A, B, C fixed and D variable.

$$\begin{aligned}\Sigma OL^2 \sin 2A &= OG^2 \Sigma \sin 2A + \Sigma GL^2 \sin 2A\\ &= OG^2 \Sigma \sin 2A + \Sigma Gl^2 \sin 2A\\ &= \Sigma Ol^2 \sin 2A\\ &= R^2 \Sigma \sin 2A - Bl\,.\,lC \sin 2A - Bm\,.\,mD \sin 2B\\ &\qquad + Cn\,.\,nD \sin 2C.\end{aligned}$$

But $Bl\,.\,lC \sin 2A = 2AB\,.\cos B\,.\,AC\,.\cos C \sin A \cos A = 8R^2\,.\,\Pi \sin A\,.\,\Pi \cos A$,

$$\begin{aligned}Bm\,.\,mD \sin 2B &= 2AB \cos ABD\,.\,AD \cos ADB\,.\sin B \cos B\\ &= 2AC\,.\cos ACD\,.\,AD \cos ADn\,.\sin C\,.\cos C\\ &= Cn\,.\,nD \sin 2C.\end{aligned}$$

Thus $\Sigma OL^2 \sin 2A = 4R^2 \Pi \sin A (1 - 2\Pi \cos A)$. E. P. ROUSE.

181. [L¹. 2. b.] *A solution, not by elliptic functions, is wanted of the following:* Given 5 lines *abcde* in a plane, it is known that the pairs of points

ab, *ce* ; *bc*, *da* ; *cd*, *eb* ; *de*, *ac* ; *ea*, *bd*

are in a collineation. Prove that the fixed triangle of this collineation is self-polar as to both the conic on *ab*, *bc*, *cd*, *de*, *ea* and the conic on *ac*, *ce*, *eb*, *bd*, *da*. J. MORLEY.

182. [K.] *Proofs of Euler's Theorem, etc.*

Lemma:—The resultant of successive reflexions in two planes intersecting at an angle α and in a line l is a rotation through 2α about l.

To find the resultant of a rotation through α about an axis OA followed by a rotation through β about an axis OB.

Take OC such that the angle between the planes OAB and $OAC = \frac{1}{2}\alpha$ and the angle between OBC, $OBA = \frac{1}{2}\beta$. Then the required resultant $\equiv$ the resultant of successive reflexions in OAC, OAB, OBA, $OBC \equiv$ the resultant of successive reflexions in OAC, $OBC \equiv$ a rotation about OC through twice the angle between OCB, OCA.

To reduce any displacement of a rigid body to its simplest form.

(1) Let A, B, ... be the initial and α, β, ... the final positions of a set of parallel planes. Then it is readily shown that A and α, B and β, ... meet on a fixed plane p. Let A', B', ... and α', β', ... be the initial and final positions of a second set of parallel planes, and let p' be the locus of the intersections of A' and α', B' and β', ... Let S be any point on the intersection l of p and p' and let M, μ, M', μ' be the planes through S parallel to A, α, and A', α'. Let l and S be moved by the displacement to λ and σ. Since S lies on M and M', σ lies on μ and μ'. Hence every point of l is moved parallel to the intersection of α and α', and therefore l and S are moved to λ and σ either (i) by a translation, or (ii) by a rotation through some point O on l. In either case the displacement of the body is completed by a rotation about λ. Hence the total displacement is equivalent (i) to a translation followed by a rotation, *i.e.* in general to a screw; (ii) to two rotations about lines through O, *i.e.* to a rotation.

(2) Let two points of the body whose final positions are A' and B' be brought to A and B by a translation which brings any one point to its final position O. Let β be the reflexion of B in the plane p bisecting AA' at right angles, and let the plane q bisect $B'\beta$ at right angles. Then p passes through O since $OA = OA'$, and q passes through O and A' since $O\beta = OB = OB'$ and $A'\beta = AB = A'B'$. Now the points O, A, B are brought to O, A', B' by successive reflexions in p and q, *i.e.* by a rotation about the intersection of p and q. But three points determine the position of a body, and therefore the displacement is equivalent to a translation followed by a rotation.

These proofs can be modified to suit the case in which the initial and final positions of a figure are enantiomorphous. HAROLD HILTON.

NOTICE.

On the occasion of the Fourth International Congress of Mathematicians which will be held in Rome in 1908, an international prize for Geometry will be awarded by the *Circolo Matematico di Palermo* In honour of its founder, Professor Guccia, it will be called the Guccia medal, and will consist of a small gold medal and the sum of 3000 francs Since the Steiner prize of 1882 was awarded the theory of gauche algebraical curves has not received the attention it deserves There is in fact quite a number of important problems connected with algebraical curves, such as their classification, the study of canonical curves of given deficiency, and the like, which have yet to be solved And but few theorems in which there is a limitation either to a real field or to a given domain of rationality are known with respect to algebraical gauche curves The Guccia medal will be awarded to the author of the most important memoir making an advance in the theory of these curves No other conditions are prescribed either as to the methods of research or as to the problems to the investigated Should nothing be presented worthy of note, the prize may be awarded to the author of a memoir making an important advance in the theory of algebraical surfaces or of other algebraical manifoldnesses

The memoirs must be in Italian, French, German or English, and, with the exception of the formulae, must be typewritten They must not have been previously published Three copies inscribed with a motto must reach the President of the *Circolo Matematico di Palermo* before July 1st, 1907, accompanied by a sealed envelope containing the motto, name, and address of the author The prize winner will receive 200 free copies of the memoir, which is to be printed in one of the publications of the *Circolo* If the prize is not awarded for a new memoir, it may be given to the author of a published work on the theories mentioned, provided that it appears between the date of this notice and July 1st 1907

The "judges" are Professors Noether, Poincaré, and Segre

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BOOKS, ETC., RECEIVED.

Aufgabensammlung zur Analytischen Geometrie der Ebene. By O. Th. Bürklen. Pp. 196. 80 pf. 1905. (Göschen, Leipzig.)

Parallelperspektive. Rechtwinklige und schiefwinklige Axonometrie. By J. Vonderlinn. Pp. 112. 80 pf. 1905. (Göschen, Leipzig.)

Auslese aus meiner Unterrichts- und Vorlesungspraxis. By H. Schubert. Vol. II. Pp. 218. 4 m. 1905. (Göschen, Leipzig.)

A College Algebra. By W. B. Fine. Pp. viii, 595. 6s. 6d. 1905. (Ginn & Co.)

Radio-Activity. By E. Rutherford. pp. xiv, 580. 2nd edition, revised and enlarged. 12/6 net. 1905. (Cam. Univ. Press.)

Association of Teachers of Mathematics in the Middle States and Maryland. Bulletin No. 2. July, 1905. pp. 80. (Published by the Association.)

Vorlesungen über technische Mechanik. By A. Föppl. Vol. I. *Einfuhrung in die Mechanik.* pp. xvi, 428. 10 m. Vol. III. *Festigkeitslehre.* pp. xvi, 434. 12 m. 3rd edition of each. 1905. (Teubner.)

Elektromagnetische Theorie der Strahlung. By M. Abraham. Vol. II. pp. x, 404. 10 m. 1905. (Teubner.)

Anfangsgründe der Darstellenden Geometrie für Gymnasien. By F. Schütte. pp. 42. 80 pf. 1905. (Teubner.)

Elemente der Vektor-Analysis. By A. H. Bucherer. pp. viii, 103. 2 m. 40. 1905. (Teubner.)

Guido Hauck. By E. Lampe and A. Parisius. pp. 32. 1 m. 1905. (Teubner.)

Die Planimetrie fur das Gymnasium. Vol. I. 2nd edition. By G. Holzmüller. pp. viii, 240. 2 m. 40. 1905. (Teubner.)

The Continuum as a type of Order: An Exposition of the Modern Theory. With an Appendix on *The Transfinite Numbers.* By E. V. Huntington. (Reprinted from the "Annals of Mathematics," July and October, 1905.) 50 c. (Harvard Univ.)

The Bolyai Prize. By G. B. Halsted. (Reprinted from "Science," Sept. 1st, 1905.)

The Annals of Mathematics. Series 2. Vol. VII. No. 1. Oct. 1905.

Concerning Green's Theorem and the Cauchy-Riemann Differential Equations M. B. Porter. *On the Singularities of Tortuous Curves.* P. Saurel. *On the Twist of a Tortuous Curve. The Continuum as a Type of Order* (continued). E. V. Huntington. *A Problem in Analytical Geometry with a Moral.* M. Bôcher.

Die Anfangsgründe der Differentialrechnung und Integralrechnung. By R. Schröder. pp. vii, 131. 1 m. 60. 1905. (Teubner.)

Einleitung in die Funktionentheorie. Vol. II. By O. Stolz and J. A. Gmeiner. pp. viii, 243-598. 9 m. 1905. (Teubner.)

Encyklopadie der Elementaren Geometrie. By H. Weber, J. Wellstein, and W. Jacobsthal. Vol. II. pp. xii, 604. 12 m. 1905. (Teubner.)

Lehrbuch der Analytischen Geometrie. Vol. I. *Geometrie in den Grundegebilden erster Stufe und in der Ebene.* By L. Heffter and C. Koehler. pp. xvi, 526. 14 m. 1905. (Teubner.)

Œuvres de Charles Hermite. Edited by E. Picard. Vol. I. pp. xl, 498. 18 frcs. 1905. (Gauthier-Villars.)

Sur les Systèmes Triplement Indéterminés et sur les Systèmes triple-orthogonaux. By C. Guichard. pp. 95. 2 frcs. 1905. (Gauthier-Villars.)

Cour d'Analyse Mathématique. Tome II. By E. Goursat. (*Théorie des Fonctions Analytiques; Equations Différentielles; Equations aux Dérivées partielles; Eléments du Calcul des Variations.*) pp. vi, 640. 20 frcs. 1905. (Gauthier-Villars.)

Correspondance d'Hermite et de Stieltjes. Edited by E. Baillaud and H. Bourget. Vol. II. pp. vi, 464. 16 frcs 1905. (Gauthier-Villars.)

Book-keeping by Double Entry. By J. W. Walmsley. pp. xii, 252. 1905. (Hodgson.)

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THE TEACHING OF MECHANICS.

IN its revised form the Report of the M.A. Committee on the Teaching of Elementary Mechanics strikes me as being on the whole an exceedingly good and helpful one, though of course open to criticism in some of its details. On two or three points relating to Kinetics, I offer these remarks:

1.

Average velocity during a given interval would be better defined as 'that imagined uniform velocity with which the actual displacement would be accomplished during the given interval.' This of course implies that we have already defined *uniform velocity* (preferably as that of a body moving so that the displacement is proportional to the time taken). Then the formula: Measure of velocity = measure of displacement ÷ measure of time-interval, having been proved for *uniform velocity,* would apply at once to *average velocity* in motion of any kind.

The notion of the *average* as being that which, substituted throughout for the variable, would give the *same result,* is perhaps the fundamental one, and is certainly one which can be clearly grasped by the mathematical tyro. The average height of seven mountains, *e.g.* is the height of one of seven equally high mountains which would give the same total height when piled on the top of one another as would the seven unequal mountains. Of course the word *result* in the general statement is vague: but the appropriate meaning is obvious in most common cases of *average.*

I would also suggest that the distinction between the terms *velocity* and *speed* ought to be explained, and that in connection with the former we ought to speak of *displacements accomplished* rather than of *distances described.*

2.

Centrifugal force will never be abandoned by practical dynamists; so we had better keep it in the theory. It is as good as the term *sunrise*, which is equally incorrect from one point of view. Theoretical dynamists had better explain 'centrifugal force' rather than reject it. And the explanation can be made quite clear if we treat it as a particular case of 'reversed effective force' (to use an old phrase). If a body of mass m lbs. has an acceleration of a feet per second per second, the resultant force on it must be $m \times a$ poundals; *i.e.*, it would be in equilibrium if a force of $-ma$ poundals acted on it in addition to the actual forces. This imaginary force then, along with the actual forces, will satisfy the conditions of an imaginary equilibrium. This mode of explanation removes somewhat of the paradoxical character of 'centrifugal force' by classing it with like things, and also prepares the way for a subsequent understanding of 'D'Alembert's Principle.' The importance of the 'reversed effective force' is that, though imaginary, it is equal in all respects to the real reaction of the force that 'balances' it in the imaginary equilibrium; for example, in the case of the stone swung round at the end of a string, under no external forces, it is equal to the pull which the stone exerts on the string; or in the case of the piston near the end of its stroke, the extra stress in the piston rod due to negative acceleration of the piston rod is exactly equal to the 'reversed effective force on the piston.' Such 'inertia reactions' are becoming more important to engineers as the speeds of engines increase, as can be seen by perusing recent technical publications on vibrations and balancing of engines. And it seems to me that school mechanics ought to give a basis for understanding the language and ideas of those who apply the science, as well as an introduction to dynamics as a pure science.

3.

Impulse is not referred to in the Report. I should have liked to see a recommendation that the *impulse of a force during a given interval* should be defined as *the change of momentum it produces*, rather than (as in some books) *the product of Force and interval of time*, since the former definition, unlike the latter, applies at once whether the force be uniform or not. Besides, the former definition seems to suggest the fundamental idea better. In connection with the *Third Law of Motion*, the term *Stress* ought to be introduced, and defined after the manner of Clerk Maxwell to be the totality of that phenomenon of which the *Action* and the *Reaction* are one-sided aspects. The fact that every force implies the existence of a body acting, as well as a body acted upon, might be emphasized here.

R. F. MUIRHEAD.

ON A BINOMIAL APPROXIMATION.

THE simple approximation for the rth root of a number which is nearly an exact rth power, given by Mr. Whipple last month, is found as a rule in Bonnycastle's *Arithmetic*. Perhaps the most elegant form is this:

If a and b are nearly equal, then the nth root of a/b is approximately

$$\frac{(n+1)a+(n-1)b}{(n-1)a+(n+1)b}. \quad \text{.........................}(1)$$

This is of course the same as

$$(1+x)^{\frac{1}{n}}=\frac{2n+(n+1)x}{2n+(n-1)x}, \text{ nearly. } \text{..................}(2)$$

For example, let $a=2$, $b=1$, $n=3$, then a first approximation to the cube root of 2 is 1·25. Making this the basis for a second, we have

$$2^{\frac{1}{3}}=\frac{5}{4}\cdot\left(\frac{128}{125}\right)^{\frac{1}{3}},$$

and applying the rule to 128/125 we get at once the approximation 635/504 or 1·2599206..., which is correct to 6 decimal places.

By ordinary expansion of both sides in (2), we find the difference to be $x^3(n^2-1)/12n^3$. In our second case $x=3/125$, $n=3$, and the error is nearly ·00000042.

It can be proved by a fairly long induction, or otherwise, that this result is a particular case of the more general theorem

$$(1+x)^n=\left\{1+\frac{(n+r)r}{2r}x+\frac{(n+r)(n+r-1)r(r-1)}{2r(2r-1)1\,.\,2}x^2+\dots\right\}$$
$$\div\left\{1-\frac{(n-r)r}{2r}x+\frac{(n-r)(n-r+1)r(r-1)}{2r(2r-1)1\,.\,2}x^2-\dots\right\},$$

where the factor $r-r$ in the two $(r+2)$th terms causes evanescence of the numerators until at the $(2r+2)$th term the denominators also have the vanishing factor $(2r-2r)$ and the terms are restored.

This is an exact theorem of which a mere general case will be found in his *Methodus nova integralium valores per approximationem inveniendi*, §§ 40–42.

This note appeared, in substance, in the *Proceedings of the Edinburgh Mathematical Society*, vol. x., p. 23

J. E. A. STEGGALL.

JAPANESE MATHEMATICS.

The *Jahresbericht der Deutschen Mathematiker-Vereinigung* has lately published an address given by Professor Harzer on "The Exact Sciences in Old Japan." At first sight it seems difficult to see why the celebration of the birthday of the Emperor of the Germans should have been selected for the delivery of an oration on this topic. The Professor is himself an astronomer, so that it is interesting to note that he brushes aside as of little value anything that Japan has accomplished in the pursuit of his favourite science. Nor does he discover any ground for supposing that anything will be accomplished along this line by those whose sudden prominence in other realms has given the Western world so furiously to think. The history of the science of mathematics in the Island Empire is full of paradoxes. In the days of Pythagoras, it will be remembered, the secrets of the science were entrusted to what was to all intents and purposes a Masonic body, bound by oath not to divulge the mysteries in which they participated. And this body had a further curious and unexpected resemblance to the Masonic order as seen among the Latin races, inasmuch as its intrusion into politics led to more than one popular outbreak. In Japan the parallel almost holds good, save for the utilisation of the society for political purposes. The elementary parts of the science were open to all. He who ran might read. But before the student was permitted to penetrate into the arcana he was compelled to join a brotherhood and to take an oath of secrecy. Things have, of course, considerably changed since those days. The powers that be have become more and more convinced of the paramount necessity of improvement in the direction of mathematical training. And with the courage and directness which is such a marked characteristic of this astonishing race, they refuse to allow entrance to any of the higher courses at the University or higher technical institutions to those who have not the necessary preliminary mathematical equipment. We shall be very much mistaken if the results of these drastic measures do not make themselves very evident in the course of the next half-century.

The feeling for mathematics in Japan is of no new growth. In the royal library of Tokio there are no fewer than 2,000 mathematical works, some of them going back to the sixteenth century. The essential characteristic of the Japanese mind is a devotion to matters of practical value. This accounts for the lines on which their development took place. First and foremost among the practical questions which attracted their investigations was one of the three problems of antiquity, that of the "squaring of the circle," as it is generally called. To the discovery of the ratio of the circumference of the circle to its diameter they devoted considerable attention, and with a large measure of success. The history of the search for this ratio, the Ludolphian number (so named after Ludolf van Ceulen, who gave up his life to its discovery and succeeded in calculating it to 333 places of decimals) is practically the early history of mathematics itself. Even the geometrical proof that the number is incommensurable, given by the great Newton in his *Principia*, leads to a result which is not always true, so that his very condensed argument cannot be flawless. The best way of getting an approximate result is by means of infinite series, and these series are obtained by the use of the binomial theorem for fractional exponents. The astonishing thing is that this method was utilised by the Japanese, so that the theorem, in which Leibniz found to his disgust in 1676 that he had been forestalled by Sir Isaac Newton, was known about the same time to the Japanese, and must therefore have been discovered by them independently. It is also pretty certain that they also discovered independently the infinitesimal calculus, another of Newton's triumphs. But whatever may have been the cause, they appear to have achieved nothing of any importance after the seventeenth century. In Europe the progress of the higher mathematics proceeded apace. But the Far East possessed no minds of the order of a Euler or a Gauss. So the interest in the higher branches gradually waned and presently fell into abeyance altogether. The tide is beginning to turn, and the chances are that the future may have for us its surprises. Even the war did not stay the gifted few who are engaged on mathematical research. From Russia and from Tokio alike there has been no check in the regular appearance of the mathematical and physical periodicals with their papers and memoirs. A nation that once produced its Newton may yet produce another.

And what about the neighbours of Japan? It used to be supposed that the Chinese had made great progress in theoretical mathematics. This was due to a misapprehension on the part of a Jesuit missionary who did not know that the sole duty of the important department known as the Board of Mathematics was to prepare an annual almanac with its dates and predictions, and not to promote the study of mathematics throughout the Empire. But even while we are writing we hear that there is promise for the future in China. The following theorem was recently sent to a Japanese mathematician by one of his Chinese friends.

Let *ABCD* be a convex in-quadrilateral to a circle of centre O; 2α, 2β, 2γ, 2δ the smaller arcs subtended by the chords *AB*, *BC*, *CD*, *DA*; r_a, r_b,

r_c, r_d the in-radii of the triangles *BCD*, *CDA*, *DAB*, *ABC*. Then since in any triangle we have $r=4R\Pi\sin\frac{A}{2}$, we have at once

$$r_a=4R\sin\beta\sin\gamma\sin(\alpha+\delta)\,;\quad r_c=4R\sin\alpha\sin\delta\sin(\beta+\gamma),$$

whence $$r_a+r_c=4R\Sigma\sin\alpha\sin\beta\sin\gamma\cos\delta,$$

and by symmetry $$=r_b+r_d.$$

Now consider a polygon $A_1A_2\ldots A_n$ inscribed in a circle, and let S_r be the sum of the in-radii of the triangles which have A_r for a common vertex and the sides of the polygon (except those on which A_r lies) as sides opposite A_r.

Then shall $S_1=S_2=\ldots=S_{n-1}=S_n$.

For if $S_1=S_r$ in the polygon $A_1A_2\ldots A_r\ldots A_nA_1$, then by analogy $S_1'=S_r'$ in the polygon $A_1A_2\ldots A_r\ldots A_nA_{n+1}A_1$.

This is clearly seen, because

$$S_1'=S_1+r_1\,;\ S_r'=S_r+r_2+r_3-r_4,\ \text{where } r_1, r_2, r_3, r_4$$

denote the in-radii of the triangles $A_1A_nA_{n+1}$, $A_rA_nA_{n+1}$, $A_rA_{n+1}A_1$, $A_rA_nA_1$. But in the quadrilateral $A_rA_nA_{n+1}A_1$, we have $r_1+r_4=r_2+r_3$; $\therefore\ S_1=S_r$, and it follows that $S_1'=S_r'$. (*Mathesis*, Dec., 1905.)

This resurrection of Chinese mathematics reminds us of the poet's lines:

> "She let the legions thunder past
> And plunged in thought again."

W. J. G.

REVIEWS.

Œuvres de Charles Hermite. Publiées sous les auspices de l'Académie des Sciences, par Émile Picard. (Paris: Gauthier-Villars, 1905). Tome I. Pp. xl, 498.

The appearance of this volume is very welcome, and its successors will be eagerly awaited. The editor, assisted by the late Prof. Xavier Stouff, has corrected various misprints, 'sometimes at the cost of long calculations'; this, and the addition of some brief notes, will save the reader much trouble, and occasional risk of error (*e.g.* on p. 163 there is a correction of a casual mistake in one of Hermite's letters to Jacobi).

This volume contains, amongst other things, the memoirs on quadratic forms, on homogeneous binary forms, and on the theory of the transformation of Abelian functions. Apparently the arrangement is chronological, but although the source of each paper is given, the date is not always stated, and this is somewhat of a blemish in an edition of this kind. Possibly a dated list is reserved for the final volume.

By way of preface the editor has reprinted the admirable lecture on the scientific work of Hermite which he delivered at the Sorbonne in 1901. The frontispiece is a portrait of Hermite at the age of about 25, very youthful, somewhat roguish, but with solid brow and brooding eyes. In size the book is convenient, being a large octavo, and it is as needless to commend the printing as the competency of the editor.

G. B. MATHEWS.

Sur les Systèmes triplement indéterminés et sur les Systèmes triple-orthogonaux. Par C. Guichard (*Scientia*, June, 1905). **Paris:** Gauthier-Villars, 1905. Pp. 96.

The object of this memoir is indicated in the title; the systems considered are of points, lines, and planes, and the work is a development of the author's previous investigations (*Ann. de l'Ec. N.* '97, '98, '03). The method is of the school of Darboux. The principal feature of novelty is the use made of certain orthogonal determinants connected with the system discussed. Various interesting results are obtained and problems suggested; to explain them here would take too much space, but those interested in partial differential equations or infinitesimal geometry will find this memoir very attractive. G. B. MATHEWS.

Abstrakte Geometrie. BY KARL THEODOR VAHLEN. (Teubner, Leipzig, 1905.)

This book is not a text-book but an account of the investigations of its author in the domain of the Axioms of Geometry, Euclidean and, so-called, non-Euclidean. Though it claims to be complete in itself, we venture to think that the book would be almost unintelligible to anyone who was not somewhat deeply read in the recent German and Italian literature of the subject. Moreover, published as it is in book form and not in the Proceedings of a learned Society, it has presumably escaped the criticism of referees. In any case, though it is attractively got up, and contains much suggestive matter, it cannot be regarded in its present form as presenting that "strictly logical and flawless system of axioms and postulates for each of the three possible geometries," the construction of which is stated in the preface to be the main object the author has had in view. Coming to details, the uninitiated English reader will be at once struck by the free use in the preliminary portion of the work of the ideas and nomenclature of the theory of sets of points. This, the youngest of all branches of mathematical science, is fast extending its sway over all the other branches. The next portion of the work is devoted to projective geometry, and the attempt is made to base this branch entirely on (1) the single idea of point as fundamental concept, and (2) the fact that there is at least one point A and at least one point B other than A as single fundamental axiom, thus avoiding in appearance at least the introduction, for example, of the straight line as fundamental concept. Here, as may be supposed, the object is in the main not to obtain new results, but to show how the axioms assumed lead to facts of common knowledge. The reasoning is often rather involved and though symbols are freely used the proof of a single proposition often occupies many pages. In the first instance, the axioms of order and continuity are avoided, as well as that of parallels. As examples of the results so obtained, we may cite the theorem that the necessary and sufficient condition that a plane geometry should be the section of a space geometry is that the theorem of Desargues should hold good, a result proved previously by Hibbert, but only on the assumption of the truth of the parallel axiom. The Pascal theorem is then proved by means of the axioms of order and continuity.

The remainder of the book is devoted to "Affine" geometry and metrical geometry. We should like in conclusion to put in a plea for the overworked modern mathematician. Every memoir is supposed to have prefixed to it a full account of the contents of the paper, pointing out what parts are new either in matter or mode of treatment. This useful rule is only too often transgressed, and notably so in the present instance. The only account of the subject matter, which occupies three hundred pages in all, is contained in barely two pages of preface. In an ordinary text-book a table of contents sufficiently supplements the preface, but in the case of a book such as the one before us we think much more was required. W. H. YOUNG.

Correspondance d'Hermite et de Stieltjes. (Gauthier-Villars, 1905.)

These two volumes contain 432 letters, extending from 1882 to the death of Stieltjes in 1894; the appendix consists of 4 letters from Stieltjes to Mittag-Leffler. The letters had been prepared for the press in Hermite's lifetime, and it appears that Hermite had promised to write an introduction to them; but this duty has been left to M. Picard (Hermite's son-in-law). The editors are MM. Baillaud and Bourget, colleagues of Stieltjes at the University of Toulouse; they have added a portrait and memoir of Stieltjes, two portraits of Hermite and a facsimile of Stieltjes' manuscript, as well as numerous notes scattered through the book.

The correspondence throws many interesting side-lights on the work of the two writers; it shows, for instance, how Stieltjes constantly tested theoretical results by arithmetical calculations, even at the cost of great labour. This practice was doubtless a consequence of his astronomical training in the Observatory at Leyden (1877-1883). It is noteworthy too, as Picard says in the Preface, "que sous cette forme plus personnelle le langage abstrait de l'Analyse perde de sa sécheresse et que la Mathématique y devienne plus humaine."

An adequate description of the results given in the letters is impossible here, owing to lack of space. But it may be noted that Stieltjes began his work with investigations on the Theory of Numbers, which led him to Riemann's ζ-function, and thence he passed to more general function-theory. His investigations on Continued Fractions seem to date from the beginning of 1887, so far as can be seen from the letters; this was shortly after his appointment (October, 1886) to Toulouse; and he continued working at the theory as long as strength remained.

A few elegant results, taken almost at random, may be mentioned. Letter 130 (October, 1888) contains the beautiful integral of Euler's differential equation,

$$\frac{dx}{X^{\frac{1}{2}}}=\frac{dy}{Y^{\frac{1}{2}}},\quad X=(a_0,\, a_1,\, a_2,\, a_3,\, a_4 \between x,\, 1)^4,$$

namely:

$$\begin{vmatrix} a_0, & a_1, & a_2-2c, & 1 \\ a_1, & a_2+c, & a_3, & -\frac{1}{2}(x+y) \\ a_2-2c, & a_3, & a_4, & xy \\ 1, & -\frac{1}{2}(x+y), & xy, & 0 \end{vmatrix}=0$$

where c is the arbitrary constant. Letter 290 investigates Klein's theorem on the number of zeros of the hypergeometric series between 0 and 1; the method seems to be not essentially different from Hurwitz's (*Math. Annalen*, Bd. 38). In letter 393, Stieltjes shows that even if the sum of an infinite series is arbitrarily small for *all* real values of x, yet it cannot be deduced that *all* the coefficients are correspondingly small. Letter 399 contains a striking result on uniform convergence, which is reproduced in facsimile of Stieltjes' writing at the end of Vol. II.

It is curious to see that Hermite felt the burden of University Examinations. In letter 5, he says, "Ce sont les devoirs universitaires, qui surchargent les pauvres géomètres et entravent leurs recherches. À la Sorbonne nous avons maintenant une session d'examens de baccalauréat et j'ai le regret de passer bien du temps à lire des compositions et à interroger sur l'Arithmétique la Géométrie élémentaire, etc."; and again (in 247) "Je ne puis vous dire à quels efforts je suis condamné pour comprendre quelque chose aux épures de la Géométrie descriptive, que je déteste, et à des choses comme la formule des annuités en Arithmétique, etc." He envies, too, the German professors who can hand over to their pupils the work of completing the calculations encountered in their researches; and says (in 418) "Ces calculs, je les ferai moi-même, en me trompant et les recommençant, en invoquant vainement, je le crains bien, la devise de Guillaume d'Orange."

Doubtless many English workers will sympathise with the feelings of the great Frenchman. T. J. I'a. B.

Sitzungsberichte der Berliner Mathematischen Gesellschaft (Third Year, 1904). Extracted from the Archiv f. Math. u. Phys. (3), Bd. 7, 8. (Teubner.)

This volume of 85 pages contains several papers of interest, amongst the results in which the following may be mentioned:

Kneser gives an elegant proof (independently of the theory of Fourier's series) that if

$$\int_0^{2\pi} f(x)\cos mx\,dx = 0, \quad \int_0^{2\pi} f(x)\sin mx\,dx = 0, \quad (m = 0, 1, 2, 3, \ldots)$$

then $f(x)$ is identically zero.

Koppe has a note on the base of Neper's (Napier's) table of logarithms. It has been stated by many writers that Neper attached the logarithm $n\lambda$ to the number $(1-\lambda)^n$; λ being used here for 10^{-7}. As a matter of fact Neper's logarithm was the arithmetic mean between $n\lambda$ and $n\lambda/(1-\lambda)$. An easy calculation shows that Neper's base is effectively $(1/e)(1-\frac{1}{8}\lambda^2)$, if λ^3 is neglected; and thus Neper's tables (when freed from certain numerical errors) really give the natural logarithms to 14 decimals. It is interesting to observe that Neper's definition is, in modern notation, $\log r = \int_r^1 dx/x$; a definition which has been recently brought forward as a basis for the analytical theory.

Lampe gives a collection of solutions to harder problems on maxima and minima; No. 6 is:—To draw through a tangent to the (circular) base of a right cone, an elliptic section whose area is a maximum or minimum. This problem deserves mention, because the solutions sometimes given are incomplete (*e g.* Williamson's *Differential Calculus*, Art. 148: Frenet's *Exercises*, 216).

Schafheitlin gives a theorem on Bessel functions. It was suspected by Nielsen that two equations $J_m(x)=0$, $J_n(x)=0$ could not have a common root (other than $x=0$), supposing the difference $(m-n)$ to be an integer; it is here proved that with $m=n+3$, and with a special value for n (nearly equal to 4) the second root of J_n is the same as the first root of J_m.

T. J. I'a B.

Introduction to the Infinitesimal Calculus. By H. S. Carslaw. Pp. 103. (Longmans, Green & Co., 1905.)

Kurze Einleitung in die Differential- und Integral Rechnung. Von Irving Fisher. Pp. 72. (Teubner, 1904.)

These two little books are very much alike in character. Each is designed to serve as a short introduction to the Calculus sufficient for the purposes of a special class of students. Mr. Carslaw is thinking of engineering students, Dr. Fisher of students of economics, and the former naturally require a good deal more mathematics than the latter; and so, although the range of the subjects treated is much the same in the two books, Mr. Carslaw generally gives us a good deal more in the way of detailed applications. Each author cherishes a hope that his book may be useful as an introductory course even for mathematical students. In Mr Carslaw's case this hope is certainly justified. I have myself subjected his book to the best of all possible tests, actual use in teaching backward students, and find it admirably adapted for the purpose. But I am afraid there is hardly enough detail in Dr. Fisher's book to make it very useful in such cases.

A great merit of Mr. Carslaw's book is his treatment of the elements of "Conics" in the only sensible way, viz.: as an easy illustration of the processes of the Calculus. He only devotes one chapter to the Conic Sections, but a great many of their elementary properties for which there is no room in the text are introduced in admirably chosen examples. It is a very great help to have an easy book from which one can teach Analytical Geometry and the Calculus at the same time. I wish that Mr. Carslaw had seen his way to include the elements of the theory of e^x and $\log x$ as well. These functions ought only to be introduced after the notions of the Calculus have been mastered, and Mr. Carslaw might have done it all in a very few pages. He says "we assume a knowledge of the properties of the following series:—$e^x=\ldots$, $a^x=\ldots$, $\log(1+x)=\ldots$." But the kind of person who will use this book has really no such knowledge at all: at least that is my experience—and there really is no book in which exactly what is wanted can be found. It is the treatment of the exponential and the logarithm which seems to me the least satisfactory feature in both books. Both authors assume too much, and neither makes it very clear precisely what he is assuming. On the other hand both authors handle the troublesome (though of course not really difficult) subject of "differentials" in a lucid and unobjectionable way. G. H. HARDY.

MATHEMATICAL NOTES.

183. [P. 3. b.] The name of the author of the note on Inversion in the *Gazette*, No. 47, p. 88, was accidentally omitted. The note was by Mr. V. Ramaswami Aiyar, of Arni, India. For want of space we were compelled to omit certain applications and a further development of the theorem.

184. [I. 1.] *The third approximation to the n^{th} root of a number.*

Mr. C. S. Jackson and Mr. F. J. W. Whipple have given investigations (in the May and December numbers of the *Gazette*) for the approximation*

$$(1+x)^{\frac{1}{n}} = (1+px)/(1+qx),$$

where x is small and $p=\frac{1}{2}(1+1/n)$, $q=\frac{1}{2}(1-1/n)$.

The following method follows the lines which I have suggested (*Gazette*, October, 1904) for dealing with some of the elementary power-series.

Write $$y=(1+x)-[(1+px)/(1+qx)]^n,$$

then $$\frac{dy}{dx}=1-n(p-q)\frac{(1+px)^{n-1}}{(1+qx)^{n+1}};$$

so that $\frac{dy}{dv}$ vanishes with x, if $n(p-q)=1$.

Then $$\frac{d^2y}{dx^2}=\frac{(1+px)^{n-1}}{(1+qx)^{n+1}}\left[\frac{(n+1)q}{1+qx}-\frac{(n-1)p}{1+px}\right],$$

which also vanishes with x, if $(n-1)p=(n+1)q$.

Thus $p=\frac{1}{2}(1+1/n)$, $q=\frac{1}{2}(1-1/n)$, as given above; with these values we find

$$\frac{d^2y}{dx^2}=\frac{n^2-1}{2n^2}x\frac{(1+px)^{n-2}}{(1+qx)^{n+2}}.$$

If we differentiate $(1+px)^{n-2}(1+qx)^{-(n+2)}$, it will be found that the result is

$$-(1+px)^{n-3}(1+qx)^{-(n+3)}(1+4pqx).$$

Thus $(1+px)^{n-2}(1+qx)^{-(n+2)}$ decreases as x increases from 0; and consequently is less than 1 for positive values of x.

That is, $$\frac{d^2y}{dx^2}<\frac{n^2-1}{2n^2}x \text{ or } \frac{d}{dx}\left(\frac{dy}{dx}-\frac{n^2-1}{4n^2}x^2\right)<0, \ x \text{ being positive}$$

Now $\frac{dy}{dx}-\frac{n^2-1}{4n^2}x^2$ is zero for $x=0$, and so is negative for positive values of x.

That is, $$\frac{d}{dx}\left(y-\frac{n^2-1}{12n^2}x^3\right)<0,$$

and by the same argument,

$$y-\frac{n^2-1}{12n^2}x^3<0.$$

Similarly, since $\frac{d^2y}{dx^2}$ is positive for positive values of x, we infer that y is positive; so that

$$0<y<\tfrac{1}{12}(1-1/n^2)x^3, \text{ if } x \text{ is positive.}$$

---

* To reduce this to the form given, write $x=N/a^n$, and multiply both sides by a.

Since $(1+px)/(1+qx)$ is greater than 1, it now follows that *

$$0<(1+x)^{\frac{1}{n}}-(1+px)/(1+qx)<y/n\,;$$

so that *the error in the approximation (for positive values of x) is less than*

$$\tfrac{1}{12}(n^2-1)x^3/n^3,$$

provided n has any rational value greater than 1.

Mr. Jackson gave this result as an approximation to the error, but without noticing that the estimated error was really in excess of the true error, for positive values of x.

In the special cases $n=2$, 3 (which are those of chief interest, perhaps), it is easy to verify by direct multiplication that

$$(1+x)(1+\tfrac{1}{4}x)^2-(1+\tfrac{3}{4}x)^2=\tfrac{1}{16}x^3<\tfrac{1}{16}x^3(1+\tfrac{1}{4}x)^2,$$

$$(1+x)(1+\tfrac{1}{3}x)^3-(1+\tfrac{2}{3}x)^3=\tfrac{1}{27}(2x^3+x^4)<\tfrac{2}{27}x^3(1+\tfrac{1}{3}x)^3,$$

leading to

$$(1+x)^{\frac{1}{2}}-(1+\tfrac{3}{4}x)/(1+\tfrac{1}{4}x)<\tfrac{1}{32}x^3,$$

$$(1+x)^{\frac{1}{3}}-(1+\tfrac{2}{3}x)/(1+\tfrac{1}{3}x)<\tfrac{2}{81}x^3.$$

These are the same results as were found above by the aid of the calculus; but to extend the algebraic method to any integral value of n seems rather laborious, although some of the transformations are interesting.

It may be worth while to recall the fact that Dedekind makes use of this approximation (for $n=2$) in his theory of irrational numbers.

T. J. I'A. BROMWICH.

185. [A. 1. c.] *Note on a point in the demonstration of the Binomial Theorem.*

When x is numerically less than 1, the series $1+mx+\dfrac{m(m-1)}{1\,.\,2}x^2+\ldots$ is absolutely convergent for all values of m, and denoting it by $f(m)$ we establish the addition theorem that $f(m)\,.\,f(n)=f(m+n)$.

The Binomial Theorem is, that $f(m)$ is, for all values of m, equal to $(1+x)^m$; that is, to the *positive* quantity which, under the theory of Indices, is represented by the expression $(1+x)^m$.

Usually, the theorem is first shown to be true for positive integral values of m, and then it is considered for positive fractional values, and finally for negative values of m.

In dealing with the case when m is a positive fraction $=\dfrac{p}{q}$, what is usually done is, after proving that the qth power of $f\left(\dfrac{p}{q}\right)$ is equal to $(1+x)^p$, it is quietly assumed that $f\left(\dfrac{p}{q}\right)=(1+x)^{\frac{p}{q}}$. But if q be an even integer, might not $f\left(\dfrac{p}{q}\right)$ be rather equal to $-(1+x)^{\frac{p}{q}}$?

This difficulty (so far as I know) is raised and discussed only in *Chrystal*; but the discussion there is by no means elementary. The point, however, admits of an elementary treatment, by taking the cases as follows.

CASE I. If m is a positive integer, we have $f(m)=\{f(1)\}^m=(1+x)^m$, and there is no ambiguity.

CASE II. If m is a negative integer, $=-n$ say, we have

$$f(-n)\,.\,f(n)=f(0)=1\,;$$

---

* If a, b, n are all greater than 1, $a^n-b^n>n(a-b)$. This is obvious on division if n is an integer; if n is not an integer we have

$$\frac{d}{dx}(x^n-1)=nx^{n-1}>n$$

if $x>1$. Thus $x^n-1>n(x-1)$. Write $x=a/b$, and the result follows.

$$\begin{aligned}\therefore\ f(m)&=f(-n)\\ &=\frac{1}{f(n)}=\frac{1}{(1+x)^n},\ \text{by Case I.,}\\ &=(1+x)^{-n},\ \text{by the theory of Indices,}\\ &=(1+x)^m.\end{aligned}$$

CASE III. Let m be a negative quantity numerically less than unity, $=-m'$ say.

Then $$f(m)=1+mx+\frac{m(m-1)}{1.2}x^2+\frac{m(m-1)(m-2)}{1.2.3}x^3+\ldots$$ $$=1-m'x+\frac{m'(m'+1)}{1.2}x^2-\frac{m'(m'+1)(m'+2)}{1.2.3}x^3+\ldots.$$

If x be negative, each term in the series is positive, and therefore $f(m)$ is positive. If x be positive, the terms are alternately positive and negative, and each term is numerically greater than the preceding term. Hence in this case also the value of $f(m)$ is positive. Thus, whether x be positive or negative, $f(m)$ is, *a priori*, seen to be a positive quantity.

Let now $m=-p/q$ where p and q are positive integers, $p<q$.

Then $$\begin{aligned}\{f(m)\}^q&=f(mq)=f(-p)\\ &=(1+x)^{-p},\ \text{by Case II.;}\end{aligned}$$

$\therefore\ f(m)=(1+x)^{-\frac{p}{q}}$, that is, the *positive* value denoted by this expression, $=(1+x)^m$.

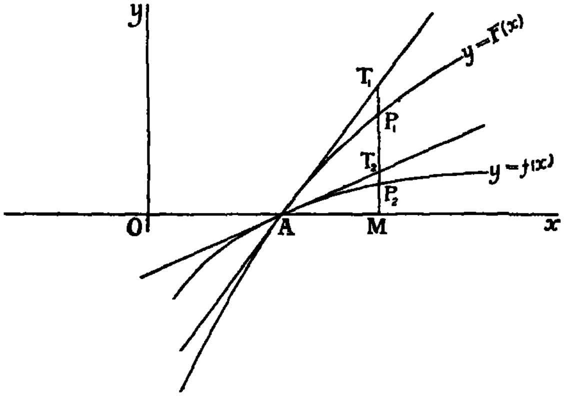
CASE IV. Lastly, let m be a quantity not falling under any of the above cases. Then we can put $m=n-m'$ where n is a positive or negative integer, and m' is positive and less than 1. Accordingly,

$$\begin{aligned}f(m)&=f(n-m')=f(n).f(-m')\\ &=(1+x)^n.(1+x)^{-m'},\ \text{by Cases I. or II., and III.,}\\ &=(1+x)^{n-m'},\ \text{by the theory of Indices,}\\ &=(1+x)^m.\end{aligned}$$

V. RAMASWAMI AIYAR.

186. [C. 1. e. α.] *Indeterminate Forms.*

A geometrical illustration.



Let $y=F(x)$ and $y=f(x)$ be the equations of two curves, each intersecting the axis of x at the point $x=a$.

The ratio of corresponding ordinates is indeterminate when $x=a$.
But, drawing the tangents AT_1AT_2 at A, we have

$$\frac{F(x)}{f(x)}=\frac{P_1M}{P_2M}=\frac{P_1M}{T_1M}\,\frac{T_1M}{T_2M}\,\frac{T_2M}{P_2M}$$

$$=\frac{P_1M}{T_1M}\cdot\frac{AM\tan T_1AM}{AM\tan T_2AM}\cdot\frac{T_2M}{P_2M},$$

and therefore when AM diminishes without limit

$$\operatorname*{Lt}_{x=a}\frac{F(x)}{f(x)}=1\times\frac{\tan T_1AM}{\tan T_2AM}\times 1$$

$$=\frac{F'(a)}{f'(a)}.$$

The extension to the case where $F'(a)$ and $f'(a)$ both vanish is obvious.

C. S. Jackson.

187. [J. 5.] *The Continuum.*

Readers of the *Gazette* may be interested to know that Professor E. V. Huntington has concluded his series of articles in the *Annals*, "Mathematics on the Continuum" and the "Transfinite Numbers." They form an *elementary* introduction to some of the problems so actively debated at the present time in the field of Cantor's *Mengenlehre*.

W. J. G.

188. [C. 1. e.] *Proof of Taylor's Theorem.*

Let $$R=f(z)-f(z-h)-hf'(z-h)-\frac{h^2}{\underline{|2}}f''(z-h)-\ldots-\frac{h^n}{\underline{|n}}f^{(n)}(z-h),$$

then $$\frac{dR}{dh}=\frac{h^n}{\underline{|n}}f^{(n+1)}(z-h).$$

Keeping z constant, $$R=\int\frac{h^n}{\underline{|n}}f^{(n+1)}(z-h)\,dh+\text{const.}$$

But $R=0$, if $h=0$;

$$\therefore\ R=\int_0^h\frac{h^n}{\underline{|n}}f^{(n+1)}(z-h)\,dh,$$

which can be transformed or discussed in the usual way. The only thing the student has to remember is to put z for $x+h$, and therefore $z-h$ for x in the usual formula.

G. H. Bryan.

189. [K. 13. a.] The *Remarque Minuscule* (Note 167, *Gazette*, May 1905) occurred to me also in 1887, and has been set in Aberystwyth and University of Wales' Examinations. It is, however, probably older, and contained in Bellavite's striking theorem, viz. if $ABCD\ldots$, $A'B'C'D'\ldots$ be similar polygons inversely situated, and if $AA'BB'CC'$, etc., be divided at P, Q, R, etc., each in the ratio of the linear dimensions of the polygons, then P, Q, R, etc., lie in a straight line.

The following extension of simple proof by Vector methods. If P_1, P_2 be points in AA' such that $AP_1\,.\,AP_2 : A'P_1\,.\,A'P_2 :: AB^2 : A'B'^2$, and if BB' be divided at Q_1Q_2 similarly to AP_1P_2A', then P_1Q_1 P_2Q_2 are equally inclined to AB, $A'B'$.

R. W. Genese.

190. [V. 1. a.]. *An apparatus for teaching long multiplication.*

I devised this, in the first instance, to enable my pupils to change from the old method (left to right) to the new (right to left), without confusion.

It consists essentially of a blackboard composed of four (or more) rectangular slats ($1{\cdot}2\times{\cdot}15$ m.'s each) which slide in horizontal grooves. The local carpenter made mine for 12s. 6d.

The following examples will show the use of this device:

Name the slats, from the top one downwards, *A*, *B*, *C*, *D*.

On *A* write any number, say 2763.

Multiply by 7; write the result, 19341, on *B* so that the 4 is directly beneath *A*'s 6.

(1) Add. Verify, result is *A* × 8. Note that (1) is "useless."

(2) Move slat *B* one place to the left. Add. Verify, result is *A* × 71. Remove *B*. Multiply *A* by 70, adding in *A* concurrently. Thus, for 2763 × 71 we have:

"3". Write 3.
"7 × 3 = 21, and 6 make 27." Write 7.
"7 × 6 = 42, carry 2; 44, and 7 make 51." Write 1.
"7 × 7 = 49, carry 5; 54, and 2 make 56." Write 6.
"7 × 2 = 14, carry 5; 19." Write 19.

Answer, 196173. Verify,

$$\begin{array}{l} \quad 2763 \times 71 \\ 19341 \\ \hline 196173 \end{array}$$

(3) Replace *B*; move it one place to the right. Add. Verify, result is *A* × 17. Remove *B*. Multiply *A* by 7, adding in 10*A* concurrently. Thus, for 2763 × 17 we have:

"7 × 3 = 21." Write 1.
"7 × 6 = 42, carry 2; 44, and 3 make 47." Write 7.
"7 × 7 = 49, carry 4; 53, and 6 make 59." Write 9
"7 × 2 = 14, carry 5; 19, and 7 make 26." Write 6.
"Carry 2; 2, and 2 make 4." Write 4.

Answer, 46971. Verify,

$$\begin{array}{r} 2763\ \times 17 \\ 19341 \\ \hline 46971 \end{array}$$

(4) Replace *B*. Extend (2) and (3) for multiplication by 701, 107, etc.

(5) Replace *B*. Multiply *A* by 2; write the result, 5526, on *C* so that the 6 is directly below *A*'s 3.

Add. Verify, result is *A* × 10 (*c*).

Place slats successively in position for multiplication by 27, 72, 271, 721, 172, 127, 217, 712, 19, 91, 73, 37, 28, 82.

E.g.

$$\begin{array}{l} 2763 \times 72\ (a) \\ 19341 \\ \quad 5526 \\ \hline 198936 \end{array} \qquad \begin{array}{l} \quad\ 2763 \times 721\ (a) \\ 19341 \\ \quad 5526 \\ \hline 1992123 \end{array} \qquad \begin{array}{l} 2763 \times 127\ (b) \\ \ 19341 \\ \ 5526 \\ \hline 350901 \end{array}$$

$$\begin{array}{l} \quad 2763 \times 712\ (a) \\ 19341 \\ \quad\ 5526 \\ \hline 1967256 \end{array} \qquad \begin{array}{l} 2763 \times 19\ (c) \\ 19341 \\ \ 5526 \\ \hline 52497 \end{array} \qquad \begin{array}{l} \ 2763 \times 28\ (c) \\ 19341 \\ 5526 \\ \hline 77364 \end{array}$$

Note that (*a*) are correct "form" and "useful," (*b*) is incorrect "form"—correct by interchanging *B* and *C*—and "useful," while (*c*) are "useless."

(6) Extend (5) for multiplication by 7021, etc., etc.

(7) Remove *C*. Move *B* one place to the left. Subtract *A*.

Thus

$$\begin{array}{l} 2763 \times (70 - 1) \\ 19341 \\ \hline 190647 \end{array}$$

Remove *B*. Multiply *A* by 70, subtracting *A* concurrently.

Thus, for $2763 \times (70-1)$ we have:
"3 from 10 leave 7. Carry minus 1." Write 7.
"$7 \times 3 = 21$, minus 1; 20. 6 from 20 leave 14." Write 4.
"$7 \times 6 = 42$, carry 1; 43. 7 from 43 leave 36." Write 6
"$7 \times 7 = 49$, carry 3; 52. 2 from 52 leave 50." Write 0.
"$7 \times 2 = 14$, carry 5; 19." Write 19.

Answer, 190647.

$$\text{Verify,}\quad \begin{array}{r} \underline{2763} \times 69 \\ 16578 \\ \underline{24867} \\ \underline{\underline{190647}} \end{array}$$

Note that (7) is harder than (2)-(6). Dull boys fail to grasp it, though they easily comprehend the rest of the system.

The board is also useful for the teaching of approximations and contracted (or Italian) long division. F. J. O. CODDINGTON.

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REPORT OF THE COUNCIL, 1905.

THE Council has very little to report concerning the year 1905. It has been very quiet, and barren of events.

The number of new members elected during the year is 26, making a total membership of 395 at the present time.

Six numbers of the Mathematical Gazette have been issued during the year, and the Council desires to express its high appreciation of the labours of Mr. Greenstreet and those who have assisted him in his editorial work.

The Report on the "Teaching of Elementary Mathematics" has been reprinted during the year, with the addition of reports on the teaching of Elementary Mechanics and on advanced School Mathematics, and also with the regulations for Geometry in the Cambridge Previous Examination. In its enlarged form the pamphlet has been distributed to all old members of the association and also to those who have been elected during the year.

After the lapse of two years the Roll of Members has been reprinted, and a proof copy lies on the table. The Council hopes that the new Roll will be ready for distribution with the March number of the Gazette.

The Association had its first dinner in the evening after the close of the last general meeting. It was a very pleasant gathering at which about 25 members and guests were present. The second dinner will be held this evening. The Council feels that the Association is deeply indebted to Mr. C. S. Jackson, who has undertaken and carried out most successfully the whole of the arrangements on both occasions.

In accordance with the Rules of the Association two members of the Council retire annually. The retiring members this year are Mr. S. Barnard and Mr. C. Godfrey. The Council nominates Mr. Roseveare of Harrow, and Mr. Whipple of Merchant Taylors' School to fill the vacancies.

REPORT OF THE COMMITTEE ON THE TEACHING OF ELEMENTARY MATHEMATICS, 1905.

THE Committee has not met during the year 1905.

In February Mr. Hawkins drew the hon. secretary's attention to the fact that the Law Society and the Surveyor's Institution still required a knowledge of Euclid in their preliminary examinations. After some correspondence the Law Society in July removed from their regulations the words "No proof can be accepted which violates Euclid's order," and substituted "Proofs other than Euclid's will be accepted and simple riders will be set." In November the Surveyor's Institution decided "to allow candidates the choice of either Euclid or geometry." During the year the Committee's reports have been collected and republished under one cover.

A. W. SIDDONS,
Hon. Sec.

AN APPEAL.

SEVERAL members of the Association expressed their conviction, at the General Meeting and at the subsequent Dinner, that there should be more articles in the *Gazette*, and more papers read at the Meetings, on the purely pedagogic side of mathematics. It is right and necessary that the general development of the various branches of our Science should not be lost sight of in an organ such as the *Gazette*, and the reader must be kept in touch by reviews with what is being done in school text-books both at home and elsewhere. But the special aims of the Association (and the object of the *Gazette* is to secure by publicity the realisation of those aims), is to *improve the teaching* of our subject in our schools. With this view the Editor is in cordial agreement. He asks the readers of the *Gazette* to come to his assistance in the matter, to propose subjects for discussion, to offer papers, to ventilate their special difficulties in a "Question and Answer" column, in short to do for the *Gazette* what no Editor can do single-handed. If this proposal is taken up with any vigour by our readers, it should be quite possible to arrange for the regular appearance in the *Gazette* of a series of Symposia, consisting of (1) articles from the pens of its members and (2) a general summing up of the *pros* and *cons* of the points at issue by an authority who combines academic distinction with teaching experience. The latter is easy to find. It is the former who as yet have not come to the front in our Association. The Council has said in the Annual Report that this last year has been a quiet year. We are not sure that this may not be meant for a piece of biting criticism, suggesting that though more remains to be done we have not done it. Be that as it may, we appeal to the readers of the *Gazette* for their cordial co-operation in providing the material for such discussions as are indicated above. Apart from our differences as to detail in the processes of teaching, there are many other topics which should be of interest to the majority as well as to special groups of our members. Those who are engaged in Preparatory Schools should be glad to expound their views of what is feasible with boys of the age with which they deal, and the Public School Teachers of Mathematics should be glad to indicate the lines along which training should proceed, and any way of improving the preparation of the boys who are ultimately to come under their care. The fairness of the papers set at the various examinations, and their strict adherence to the official syllabus may also prove a fruitful subject of discussion.

At the moment of writing we have received a letter from a prominent member of the Association and of the Council, expressing the opinion that all teachers of Mathematics are at present in a state of difficulty, more or less. The text-books (in Geometry, especially) seem to him to incline too much to practical in preference to theoretical work, and he hazards the conjecture that the majority of good teachers regard this as an evil. We imagine that the complete change that has taken place in some of the examinations will be found by many a more real "difficulty" than even the text-books, in so far as the sound teacher is independent of the text-book. There is one syllabus which is reported to inform candidates that "systematic knowledge is not expected"(!) What kind of knowledge is expected is not stated. Subjects are increased out of all bounds, and each is intended to be a mere conglomeration of scraps. Nearly all the questions are drawing or calculations. In one case the only theoretical question was taken from Book XI., and it could not be done without the aid of Book VI. Again, it is only necessary to remind those who prepare boys for Mathematical Scholarships of a case which, if "gibbeted" at the time, would have led to greater care in the future conduct of such examinations. In the questions set in the morning and afternoon papers at one of the Colleges no less than four or five gave incorrect results. Also, the way in which some examiners habitually stray from the lines of the syllabus is ever the cause of discontent among teachers.

We feel it is only necessary to call the attention of our readers to some of the ways in which the *Gazette* may be brought more into touch with every side of their daily labours to secure the suggestions for which we appeal. There is no lack of initiative to be found among the members of the various associations founded to promote improvement in the teaching of other subjects of the curriculum, though with one exception they have not the advantage of an organ for the promulgation of their opinions. And, having such an organ, it seems a pity, to say the least of it, that our members do not utilise it to the utmost for the purposes which are the primary objects of the Association.

THE EDITOR.

SOME NOTES ON CERTAIN THEOREMS IN HIGHER TRIGONOMETRY.

I ADMIRE the ingenuity of the method by which Prof. Nanson establishes the expansions of $\sin x$ and $\cos x$ as power series in x. But I must confess that his attempt to 'simplify' the 'accepted accurate' proofs of these expansions seems to me a fundamentally mistaken one. Nor can I admit that the result is at all satisfactory 'from the elementary didactic point of view.' His proof is to my mind essentially an artificial *verification*, and altogether *unnatural*, in the sense that it has no place in any natural and logical way of developing the ideas which lead to the result. I speak with diffidence as one who has had far less experience of teaching than Prof. Nanson. But I am convinced that my criticism is a fair one: and I think that many of the proofs usually given, in English text-books, of many of the theorems of 'Higher Trigonometry,' are open to fair criticism on similar grounds, when (as is too seldom the case) they are not to be condemned for 'hopeless complexity' or utter lack of force.

And if this is so, so it will and must be so long as this subject continues to occupy its present place in the mathematical course. At present, if one may judge by results, boys are taught these theorems quite prematurely and before they can have had the opportunity of acquiring anything like a sufficient grounding of general mathematical knowledge. The great majority of boys (I am speaking, of course, only of those whose abilities are up to something approaching a scholarship standard) never really grasp the meaning of them, and one of the most fascinating and instructive regions of mathematics is, from an educational point of view, practically wasted.

How are we to define what we mean by 'Higher Trigonometry'? It is, I suppose, the theory of the exponential, logarithmic, and circular functions, and of the series and products associated with them, for real and complex values of the variable. It is true that a certain part of this theory is usually lopped off and labelled as a part of 'Higher Algebra,' but the most conservatively-minded of teachers would probably admit the absurdity of this. At any rate this is what I mean by 'Higher Trigonometry,' and there seem to me to be four theorems or groups of theorems which may be said roughly to mark the stages of our progress in the subject. These are:—

(A) De Moivre's theorem for a real and rational index.

(B) The exponential theorem, that is to say the theorem that if x is real and rational one of the values of $\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots\right)^x$ is $1+\frac{x}{1!}+\frac{x^2}{2!}+\dots$, or that one of the values of e^x is $\exp. x$.

(C) The theorem that if x is real $\exp. (ix)=\cos x+i\sin x$.

(D) The factor-theorem for $\sin x$.

It is with theorem (C) that I am particularly concerned at present. But there are one or two remarks which I should like to make about the other theorems.

(A), I think, presents no difficulty, as soon as the elements of the Argand diagram are mastered. *Irrational* powers should at this stage be severely and *explicitly* left alone. Any further extensions of the theorem should in any case be regarded as corollaries of (C); and so indeed *may* (A) itself; but the more direct treatment is better put first. (B) is a direct corollary from the multiplication theorem for $\exp. x$. This last theorem should of course be proved for *all* values of x: it naturally requires some elementary notions with regard to convergence, and how far these should be insisted upon is a matter for judgment in particular cases. The theorem

$$\lim_{n=\infty}. \left(1+\frac{x}{n}\right)^n=\exp. x$$

(with whatever degree of rigour or generality it may be stated) is instructive and important, but essentially accessory and not fundamental, and to base the exponential theorem on it, although possible, is logically quite wrong. The logarithmic expansion is more difficult and may well be postponed a while. Finally (D) is *really* difficult, and the question of what to do about it is a real *crux* in the teaching of trigonometry. There is no direct and rigid proof which is at all possible for boys, and I cannot think that anyone who has had to give a proof in lectures on the theory of functions will imagine that any indirect method of procedure is likely to lead to better results. One has only to read the account of the matter in Harkness' and Morley's excellent *Introduction to the Theory of Analytic Functions* to understand that we have here to deal with difficulties which we cannot evade. Fortunately a rigid proof is in no way necessary for profitable employment of the result, and the result is easily made *plausible* from either point of view or both.

So much for (A), (B), and (D). With regard to (C) there is much more to be said, as the theorem may be approached from so many points of view.

(i) In the first place we may postpone the theorem until a knowledge of Taylor's theorem has been acquired. There is then no difficulty at all. But although I am heartily in favour of pushing the Calculus, both Differential and Integral, as far forward in mathematical teaching as can possibly be done, I hardly think this theorem need wait so long, and we certainly lose something if it does. Moreover, if we adopt this method, the theorem (C) will naturally appear as a *consequence* of the expansion-theorems for $\sin x$ and $\cos x$. This, to my mind, is an inversion of the natural logical order, and in so far an argument against this method of procedure.*

(ii) The same argument also tells against the second possible method, which is that adopted in Chrystal's *Algebra*. The principle of this method is to give an accurate proof, independent of the notions of the Calculus, of the expansions, and to deduce (C) from them. Every deference is due to so eminent an authority as Prof. Chrystal; but I cannot help thinking that, from the elementary 'didactic' point of view, this method combines every possible disadvantage.

(iii) Thirdly, we may presuppose a little more knowledge of the foundations and a little less of the superstructure of the Calculus, and we may serve up the old line of argument,

$$y=\cos x+i\sin x,\quad \frac{dy}{dx}=-\sin x+i\cos x,$$

$$\frac{dy}{y}=idx,$$

which may be made to lead to the desired result in a perfectly simple and satisfactory way. This is in substance the way followed by Jordan in his *Cours d'Analyse*, and is the way I should generally choose myself. The only point about it is that it essentially involves the ideas of y varying along a circle in the Argand diagram in such a way that dy/y is purely imaginary, and of the logarithm as a many-valued inverse of the exponential function of a complex variable.† These ideas are simple and fundamental, and should be acquired much earlier than is usually the case; and there is nothing essentially new, to anyone who understands the meaning of an integral, in the notion of $\int\frac{dy}{y}$ when y varies along any contour. And I am sure that anyone who has not yet acquired these ideas will derive more profit from the attempt to do so than from puzzling himself with theorems which are in reality applications of them: for this is the real *genesis* of (C).

---

* A modification of this method has, I find, been suggested by Prof. Bromwich (*Math. Gaz.*, vol. iii., p. 85). There is a good deal to be said for the line of proof which he adopts, but I cannot regard it as the best.

† It has to be assumed that the exponential series can be differentiated term by term. I should never scruple, at this stage, to make assumptions such as this.

(iv) Fourthly, we may follow Stolz and Gmeiner's method in their *Einleitung in die Funktionentheorie.* They presuppose less than is necessary if the preceding method is adopted.

Suppose $$\exp.\,(ix)=u+iv.$$

Then $$u^2+v^2=\exp.\,(ix)\,.\,\exp.\,(-ix)=1,$$

and the modulus of $u+iv$ is unity. Hence

$$u+iv=\cos\phi+i\sin\phi,$$

where ϕ is *some* real angle: we may suppose $0\leqq\phi<2\pi$. In particular

$$\exp.\,(i)=\cos\phi+i\sin\phi,$$

where $0<\phi<2\pi$, as exp. (i) is evidently not equal to unity. By De Moivre's theorem

$$\exp.\,(\xi i)=\cos\xi(\phi+2k\pi)+i\sin\xi(\phi+2k\pi),$$

where k is an integer. If $\phi+2k\pi=\psi$, and we equate imaginary parts, we obtain

$$\xi-\frac{\xi^3}{3!}+\ldots=\sin\xi\psi,$$

and since $$\lim_{\xi=0}\frac{\sin\xi\psi}{\xi}=\psi$$

it follows that $\psi=1$.

This method, which I had not seen before the publication of Stolz and Gmeiner's book, is very simple and there is a great deal to be said in its favour.

(v) Fifthly, we may establish the two equations

$$\lim_{n=\infty}\left(1+\frac{x+iy}{n}\right)^n=\exp.\,(x+iy),$$

$$\lim_{n=\infty}\left(1+\frac{x+iy}{n}\right)^n=e^x(\cos y+i\sin y),$$

directly and independently. This method (in conjunction with (ii)) is followed in Hobson's *Trigonometry.* As I have indicated already, I cannot consider it a good one, and it is probably more difficult than any of the others.

It will be observed that (iii) (iv) and (v) agree with one another in making the expansion-theorems for $\cos x$ and $\sin x$ corollaries of (C), whereas in (i) and (ii) this interdependence is inverted. For this reason the three latter methods appear to me to be logically preferable, and for my part I should discard (ii) and (v) and adopt (iii) or (iv), and if possible, *both,* though of course not losing sight of (i) for such pupils as are familiar with Taylor's theorem. The simplest method is doubtless (iv), but (iii) seems to me by far the most instructive. And if the objection is raised that the range of ideas required for (iii) is impossible for schoolboys, I should reply first, that, if this is so, then Higher Trigonometry, except in so far as it is concerned with *finite* series and products only, had better be postponed; and secondly, that I do not believe that it is so, and that a wider range of general ideas and less practice in detailed application of them is precisely what is wanted in elementary teaching, in Analysis even more than in Geometry. G. H. HARDY.

A NEW CUBIC CONNECTED WITH THE TRIANGLE.

THE locus of the points of contact of tangents from A to the conics of the confocal system with foci B, C is the same curve as the locus of the feet of normals from A to conics of the system. For any tangent AP to one of the conics is a normal AP to the orthogonal conic of the system through P. Hence if the coordinates of A referred to BC and the perpendicular bisector of BC be $x_0 y_0$ the equation of this locus is the eliminant of λ between

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \quad \frac{x_0-x}{\dfrac{x}{a^2+\lambda}}=\frac{y_0-y}{\dfrac{y}{b^2+\lambda}}.$$

Thus the locus is the cubic

$$[x(x-x_0)+y(y-y_0)][yx_0-xy_0]=(a^2-b^2)(x-x_0)(y-y_0).$$

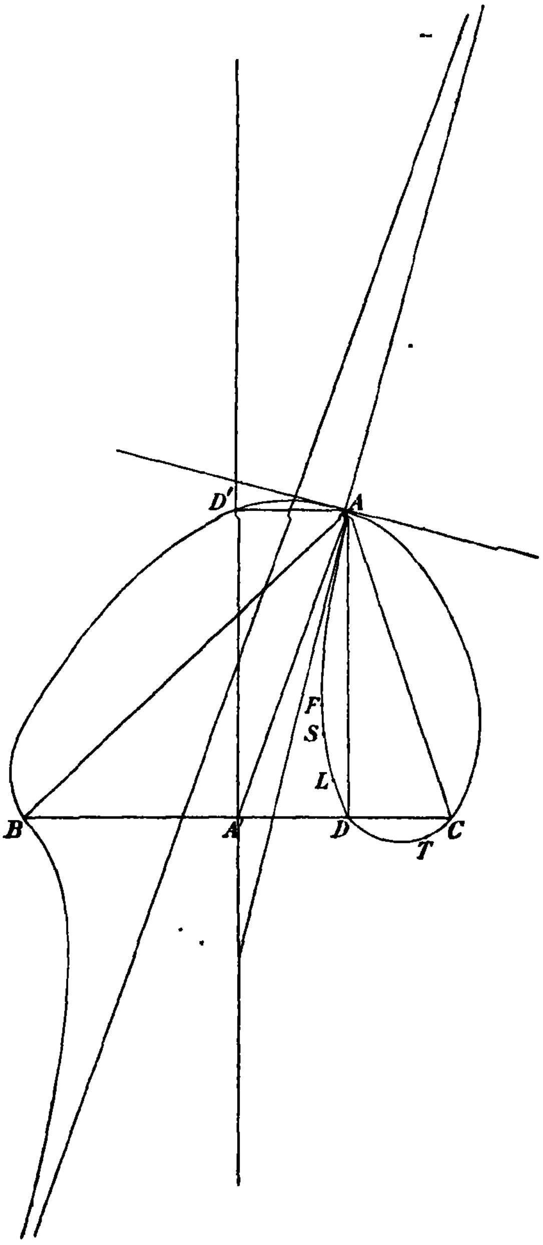
The cubic is shown in the accompanying diagram.

The points $(\pm\sqrt{a^2-b^2},\ 0)$ and $(0,\ \pm\sqrt{b^2-a^2})$ are on it, *i.e.* it passes through the real pair of foci BC of the system and the imaginary pair. It also passes through the circular points at infinity.

The points $(x_0, 0)$, $(0, y_0)$ are on it. Thus it passes through the feet of the perpendiculars from A on BC and the perpendicular bisector of BC. This can be seen geometrically since these feet DD' are the feet of normals from A to limiting conics of the system.

The point A is on it and is a node the tangents at which are the bisectors of the angle A, as is seen by considering the limiting ellipse and hyperbola through A.

Again, if P is any point on the locus, AP being a tangent to one of the conics bisects the angle BPC either internally or externally and conversely, any point having this property is on the locus. But Fermat's point, F, has this property, hence it is on the cubic. Also the focus of the parabola which touches AB, AC at B and C has this property. Hence this focus S is on the cubic.


D′
A
F
S
L
B
A′
D
C
T

The real asymptote of the cubic is

$$yx_0 - xy_0 = \text{Lt.}\,\frac{(a^2-b^2)(x-x_0)(y-y_0)}{x(x-x_0)+y(y-y_0)},$$

where $x=\frac{x_0}{t}$, $y=\frac{y_0}{t}$ and t approaches ∞.

Thus it is $$yx_0-xy_0=\frac{(a^2-b^2)x_0y_0}{x_0^2+y_0^2}.$$

Thus it is parallel to the median AA'. The intercepts on the axes are

$$-\frac{(a^2-b^2)x_0}{x_0^2+y_0^2},\quad \frac{(a^2-b^2)y_0}{x_0^2+y_0^2},$$

$$\text{i.e. } -\frac{A'C^2}{AA'^2}\cdot A'D,\quad \frac{A'C^2}{AA'^2}\cdot AD.$$

Any conic of the system is cut by the cubic in 6 points which are the feet of the four normals and the points of contact of the two tangents from A to that conic.

The external portions $AD'B$, AC of the cubic consist of the points of contact of tangents from A to those ellipses of the system for which these are real, starting with the ellipse through A and ending with the line BC. The portions $A\infty B$ and $AFDC$ consist of the points of contact of tangents from A to those hyperbolas of the system for which these are real, starting with the hyperbola through A and ending with the line hyperbola, vertices BC. Of these tangents the pair corresponding to any one hyperbola touch the same branch until one of them becomes parallel to the asymptote of the cubic, *i.e.* coincides with AA', after which they touch different branches. Now the second tangent of the pair, one of which is AA', is the isogonal line to AA', *i.e.* is AK, where K is the symmedian point. Thus if AK meet the cubic again in L, the point L has the property that the hyperbola of the system that passes through it divides those hyperbolas for which the points of contact of tangents to them lie on the same branch from those for which they lie on different branches.

Again the external portions $D'B$, $D'AC$ consist of the feet of normals from A to the hyperbolas of the system, starting with the line hyperbola $A'D'$ and ending with the line hyperbola, vertices B, C. Also the portions $\infty AFDC$, ∞B consist of the feet of the normals from A to the ellipses of the system, starting with the infinite circle and ending with the line BC. The cubic is touched at T by one of these ellipses, and all smaller ellipses cut it in 6 real points, *i.e.* have four real normals from A. For the ellipse that touches the cubic two of the four normals coalesce in AT. Thus since no other ellipse and no hyperbola can be drawn to touch the cubic except at A itself, as an examination of the figure proves, we have a theorem that only one conic of a confocal system has a point on it the centre of curvature at which is a given point A.

The figure has been drawn for the case in which B and C are acute angles. In the case of one of them being an obtuse angle, there is only a slight modification of the figure. For instance, the conic in this case which has A for the centre of curvature of a point T on it is seen to be a hyperbola.

The following table follows from an examination of the figure drawn for the case of B and C acute angles:

	Number of real Tangents from A.	Number of real Normals from A.
Ellipses from circle at infinity to ellipse through A,	0	2
Ellipses from ellipse through A to ellipse through T,	2	2
Ellipses from ellipse through T to line ellipse,	2	4
Hyperbolas from line $A'D'$ to hyperbola through A,	0	2
All remaining hyperbolas,	2	2

H. L. TRACHTENBERG.

THE PSEUDO-DEFINITION OF THE STRAIGHT LINE.

WE recognise as decadence the substitution for Euclid's theorem I. 20, "Any two sides of a triangle are together greater than the third side," of the objectionable pseudo-axiom, "A straight line is the shortest line from one point to another."

This phrase, as Hilbert has well said, is of necessity meaningless when the concept "length of a curve" has not been defined. Every book on elementary geometry which has introduced as axiom or definition of a straight line any phrase equivalent to "A straight line is the shortest distance between two points," every book which has at its beginning as axiom or definition of a straight line any phrase equivalent to "A straight line is the shortest distance between two points," has been in error. We believe the advances in regard to the foundations of elementary geometry make timely the rejection of this piece of decay from the body of the elements, and that the time is ripe for the return at this point to the pristine purity of Euclid.

If any additional argument were necessary in reference to a matter so palpable, we have it in the fact that the French themselves have repudiated this pseudo-definition introduced so unfortunately by Legendre. Of it Laisant says: "This definition, almost unanimously abandoned, represents one of the most remarkable examples of the persistence with which an absurdity can propagate itself throughout the centuries."

"In the first place, the idea expressed is incomprehensible to beginners, since it presupposes the notion of the length of a curve; and further, it is a vicious circle, since the length of a curve can only be understood as limit of a sum of rectilinear lengths; moreover, it is not a definition at all, since, on the contrary, it is a proposition or a theorem."

Reflection shows even a boy how far such a phrase is from being simple. It presupposes all the complex of assumptions necessary for measurement. But more and worse; it presumes beforehand the possibility of measuring all kinds of lines that can be drawn anywhere in space, else how could it be said that a certain one of these lines is the shortest distance between two points? Yet the straight line occurs in non-metrical, in descriptive, in pure projective geometry. All postulates demanded by metrical geometry are presupposed in this pseudo-definition. But worse; before measurement is possible we must have a standard, a unit for measurement, and the standard for length is and has always been itself the straight line. Thus measurement of length or distance presupposes the straight line as a necessary pre-condition of its own possibility, and therefore cannot be taken as a simpler and more ultimate notion with which to define the straight line, nor as a simple assumption upon which to build up the opening demonstrations of geometry. Furthermore, the very existence of a minimum is itself an assumption, and here a gratuitous and unnecessary one. These difficulties are so serious, so obvious, it is little short of a miracle that this absurd crudity has not long ago been ejected from elementary demonstrative geometry.

The common and wide-spread crude notion of the length of a string is a physical, not a geometrical notion. By making fast one end of a string and passing the other through a hole or a ring and observing how more and more of it passes out until the string becomes stretched or straight, we have an example of those experiences or experiments which confuse the physical notion of quantity of string with the geometrical notion of length of a piece of a straight line or a sect, or a combination of sects; and this physical notion is at the basis of a common mental confusion which unconsciously thinks it applicable to the length of the semi-circle. Such a mind would square or rectify the circle by putting a chalk mark on the tire of a waggon wheel, or winding a thread around a silver dollar. Such a procedure might serve to correct Solomon's value for π, but all the measurements since the world began are not enough to prove the theorem: "Two sides of a triangle are together greater than the third."

As Dr. E. W. Hobson says: "It is a very significant fact that the operation of counting, in connection with which numbers, integral and fractional, have their origin, is the one and only absolutely exact operation of a mathematical character which we are able to undertake upon the objects which we perceive. On the other hand, all operations of the nature of measurement which we can perform in connection with the objects of perception contain an essential element of inexactness. The theory of exact measurement in the domain of the ideal objects of abstract geometry is not immediately derivable from intuition."

This assumption or definition, a straight line is the shortest of lines, can therefore have no meaning until first has been determined what geometric meaning can be given to the length of a curve.

Moreover, underneath all this is the fact that even the theorem, Euclid I. 20: "Any two sides of a triangle are together greater than the third side"; a proposition which the Sophists said even donkeys knew, simple as it is, depends upon a complex of preceding assumptions. Hilbert has thrown brilliant new light upon it in the *Proceedings of the London Mathematical Society*, 1902, pages 50-68, where he creates a geometry in which two sides of a triangle may be together less than the third side, exhibiting as a specific and definite example a right triangle in which the sum of the two sides is less than the hypothenuse.

In general the length of a curve is defined by the aggregate formed by the lengths of the perimeters of a proper sequence of inscribed polygons. This definition in ordinary cases creates, fixes for the curve a length; but in case no such aggregate is convergent, the curve is regarded as not rectifiable. Even our creative definition has failed to endow it with length, and thus lengthless it remains.

If, however, it can be shown that the lengths of the perimeters of these inscribed polygons form a convergent aggregate which is independent of the particular choice of the polygons of the sequence, the curve is rectifiable, its length being defined by the number given by the aggregate. Always the recondite idea of limit has been found necessary for defining what we are to mean by length of a curve. How visible then the blunder in trying to define the straight by a metric comparison with all curves!

What geometric meaning has the phrase: "A straight line is the shortest line that can be drawn from one point to another"? A relation of equality or inequality between two magnitudes must have some foundation, and be capable of some intelligible test. In the traditional geometry the foundation of all proof by Euclid's method consists in establishing the congruence of magnitudes.

To make the congruence evident, the geometrical figures are supposed to be applied to one another, of course without changing their form and dimensions. But since no part of a curve can be congruent to any piece of a straight, so, for example, no part of a circle can be equivalent to any sect from the definition of equivalent magnitudes as those which can be cut into pieces congruent in pairs.

In any comparison of size by congruence, we must be able to place one of the magnitudes or portions of it in complete or partial coincidence with the other. No such direct comparison can be instituted between a straight and a line no piece of which is straight. Thus the whole of Euclid's Elements fails utterly to institute or prove any relation as regards size between a sect and an arc joining the same two points. The operation of measurement we cannot effect, rigorously speaking, either for curves or for curved surfaces, since the unit for length is a sect, and the unit for area the square on that sect. In fact, however little may be the parts of a curve, they do not cease to be curved, and consequently they cannot be compared directly with a sect; just as parts of a curved surface are not directly comparable with portions of a plane.

We cannot even affirm that any ratio exists between a circle and its diameter until after we have made some extra-Euclidean and post-Euclidean assumption.

Therefore when Phillips and Fisher, of Yale, give as their definition of a straight (1898, p. 4, No. 7, Def.): "A straight line is a line which is the shortest path between any two of its points," they pass through and beyond Euclid's Elements to give us his simplest element; they institute a metrical comparison not only with circular arcs, but also with all curves known and unknown; they presuppose a foreknowledge of all lines in a definition of the simplest line. Is it still needful to say this is grossly bad logic, bad pedagogy, bad mathematics?

There being no good reason for retaining longer this old bit of decadence, we desire the removal of this blemish, this parasite which has clung so persistently, so obstinately to what Sir Henry Savile called "the beauteous body of geometry."

GEORGE BRUCE HALSTED.

REVIEWS.

PRACTICAL (OR EXPERIMENTAL) MATHEMATICS.

ONE result of the criticisms on the teaching of Mathematics that originated in Professor Perry's address to the British Association in 1901 has been a great increase in the attention paid to practical applications in the elementary parts of the subject. The new regulations for the Army Examinations give considerable prominence to practical work, and under the name "Practical Measurements" the subject attains a position similar to that of Practical Work in Physics and Chemistry. The new "Leaving Certificate" Examination of the Combined Universities of Oxford and Cambridge first held in July, 1905, contained experimental work of this kind, and at one centre, if not at all, candidates were required to answer a number of questions after having made measurements and experiments on a *penny*. The value of a familiar acquaintance with actual models has long been recognised, though many teachers of mathematics used nothing more than pen and paper, or chalk and blackboard. The writer, in his three years of preparation at Cambridge for the Mathematical Tripos, remembers only one practical experiment made by a lecturer; this was the rough drawing in ink of a catenary on paper; while the ink was wet, the lecturer (a famous coach of St. John's College) rolled a chalk crayon over the figure. The experiment was to illustrate a catenary on the curved surface of a cylinder. The late R. W. H. T. Hudson was an adept in practical illustrations of mathematics. While writing his remarkable book on Kummer's Quartic Surface he employed models of his own designing, constructed out of wires, etc. In his Lecture-Room at Liverpool University he had begun to collect most interesting working models, by means of which he illustrated his lectures.

The new requirements have already produced a crop of text books.

Introductory Mathematics. R. B. MORGAN. (Blackie.)

At the beginning of his preface the author states that he has attempted "to include in one cover all the Algebra, Geometry, and Graphs a boy ought to know before he can really be said to have done more than obtain an introduction to Mathematics." What he means is clear, but somewhat strangely expressed. He states further that "he has refrained from making statements which a boy ought to discover for himself."

This looseness of expression is unfortunately somewhat common in the book, and there is a slight tendency towards exaggerated language, for instance: "It is extremely important that you should know accurately"—"Accuracy is of vital importance." There are, however, many excellent features. Elementary algebraic processes are taught by appealing to existing knowledge of arithmetic, and in this connection the rules for signs are treated admirably.

The portion on equations is good, and the author never fails to verify all solutions. There is, however, an unfortunate use of the word *proof*. Thus on p. 62: "To *prove* an identity you substitute any value you please for the same letter on both sides of the identity, simplify each side separately, and show the results to be the same."

The graphical work is very fair, but the numbers giving the scales used are not always marked on the axes. The first example based on cricket scores is not well chosen, as the number of runs is not a continuous quantity.

The geometry is not quite so well done. Triangles are not touched until after the middle of the book. In classifying quadrilaterals the author omits the kite. The Church Tower on p. 107, is an unsatisfactory example, owing to the difficulty of measuring along the ground to the point vertically beneath the top of the tower.

The experimental work is rather meagre; the author apparently has not had great experience in the practical difficulties connected with this work. For instance, to measure the circumference of a circle: "Place thread or string very carefully round the circ^{ce} so as to measure its length. If you have any difficulty in preventing the thread from slipping, use pins as you proceed." Models of solids are to be made from their "nets." Very little help is given by advice as to the difficult process of joining edges together, except by recommending the use of stamp-edging. In the construction of pyramids made from paper or cardboard, the nets given are unsuitable owing to the difficulties which occur at the apex. Perhaps it was wise of the author not to mention how hard it is to fit circular bases of paper to cylinders and cones.

The following occur in the book:

10 mm. = 200 feet.

More briefly: $1{\cdot}5'' \times 2{\cdot}3'' = 3{\cdot}45$ sq. inches.

The book is evidently the work of an experienced teacher. It is, however, difficult to see the need for it, as most of the processes may be found in books on geometry, algebra, etc., which have been recently published. Still, it may be an advantage to have all the work in a small cheap volume, such as the one above. The printing is good. No answers are given to the examples.

Practical Mathematics. A. CONSTERDINE and A. BARNES. (John Murray.)

This book is designed chiefly for industrial students, and covers elementary geometry, algebra, and arithmetic. All the work is based on the measurement of actual objects; for instance, algebra is introduced by the formula $l = \pi d$ after experimental work on the circle. The explanations, continually interlarded with little questions, will perhaps be rather bewildering to the class of student contemplated. Rules, as such, are seldom given, but suggested for discovery.

The authors take decimals before vulgar fractions, though they introduce the former by tenths, hundredths, etc. The experimental work leading to the treatment of vulgar fractions is particularly good.

Geometrical notions are carefully introduced, but the measurement of angles in degrees might be taken earlier. It is left until nearly the middle of the book. Advice to students to make preliminary "hand-sketches" is a good feature.

The definition given of a *figure* is strange: "A plane which is bounded by one or more lines."

The treatment of fractional indices would be much improved by graphs—*e.g.* of $y = 2^x$. This method forms an excellent introduction to logarithms, and it is a pity that the authors have not taken advantage of it.

In combining logarithms the vertical arrangement is more satisfactory. For instance the writers have

$$\log x = 2{\cdot}0913 + 1{\cdot}6586 + 0{\cdot}8318 = 4{\cdot}5817.$$

$$\begin{array}{rll} \text{This might be written } \log x = & \left\{\begin{array}{l}2{\cdot}0913\\ 1{\cdot}6586\\ 0{\cdot}8318\end{array}\right. & \text{or}\quad \log x = \begin{array}{r}2{\cdot}0913\\ +1{\cdot}6586\\ +0{\cdot}8318\end{array}\\ = & \;4{\cdot}5817 & \qquad\qquad\quad = \;4{\cdot}5817\end{array}$$

The theorem of Pythagoras is suggested for discovery (1) by squared paper and the Δ 3, 4, 5; (2) by construction and calculation. One or two "dissection proofs" might well have been added. In connection with this theorem and the corresponding theorems for acute- and obtuse-angled triangles too much is expected from the student, who is, moreover, asked to discover for the first time that $(a-b)^2 = a^2 - 2ab + b^2$ from a figure corresponding to that of Euc. II. 7.

The authors speak of the "Equation of a Circle," although they state that it is very important to use the words circumference and circle in their right senses.

The chapter on Plans, Elevations, etc., is instructive, and should be of great use. It is an important branch which is often neglected in schools.

Density is not defined, but the term relative density is used for specific gravity.

A chapter is given to Trigonometry. It forms a good introduction, but perhaps more attention might have been given to the graphs of the ratios. The examples in this chapter are not well chosen, as they do not give sufficient prominence to practical applications. Moreover, no instructions are given as to the use of trigonometrical tables.

The book concludes with a short chapter on Time and Velocity.

Tables of logarithms, antilogarithms, and of values of sin, cos, and tan for angles of a whole number of degrees are placed in appendices at the end. Answers to the examples are given.

The book shows much originality, and though exception may perhaps be taken to the order and arrangement, it well fulfils the aims for which it was written.

A Note-Book on Experimental Mathematics. C. GODFREY and G. M. BELL. (Arnold.)

This book is arranged on the lines of Ashford's *Note-Book of Practical Physics*, compiled for use in Harrow School, and is intended primarily to cover the requirements of the army entrance examinations.

An list of apparatus required in the experiments is given at the beginning of the book. This should prove most useful.

There are 124 experiments with brief, but generally sufficient instructions to the student. The concrete conceptions which he should gain by working through them should be of great value to him when studying the more theoretical portions of mathematics. It is important, however, that classes taking practical work should frequently discuss their work with the teacher. It is therefore advisable, though sometimes difficult to arrange, that the mathematical master should take both the theoretical and practical work of his form. This will generally necessitate a "Practical Mathematics Room" in addition to the Physical Laboratory. Such a room, unless the Physical Laboratory is used for the purpose, will therefore be necessary for forms in schools using this book. In fact at some of the larger schools a Practical Mathematics Room has already been established.

The book is divided into four parts; Part I. contains simple experiments and measurements connected with lengths, areas, volumes, mass and density, and fluid pressure. Part II. is on statics, with experiments on tensions, moments, and the equilibrium of three forces. The board used to illustrate the parallelogram of forces is placed in a vertical plane, and the three forces are obtained by using hanging weights. The method with three spring-balances is perhaps more easy to set up.

Part III. commences with instruments of more refined accuracy—the vernier, screw-gauge, spherometer. The vernier is introduced by a neat and convincing experiment, but the principle of a micrometer screw might well have been learnt after experiments with simple screws. The radius of a sphere is to be calculated from formula $2R=h+\frac{r^2}{h}$, r being obtained by calculating the radius of the circumcircle of the equilateral triangle formed by the three feet of the instrument. For approximate purposes a geometrical construction based on h and r (measured at once) is sufficiently accurate.

Part III. also contains further work in specific gravity and statics. It ends with a little dynamics—the determination of "g" by means of a lead ball falling against an oscillating bar, and also by the simple pendulum.

The work in this part is suggestive, and it is almost a pity that there is not more of it, *e.g.* illustrations of accelerations by means of some apparatus similar to that devised by Mr. W. C. Fletcher (referred to at the end of the book). Suggestions for further experiments are given in Part IV. The authors hint that the time has come for the retirement of Atwood's machine. It certainly deserves a rest from its old occupation of trying to discover "g."

Elementary Practical Mathematics. H. A. STERN and W. A. TOPHAM. (G. Bell & Sons.)

This contains the first nine chapters of a text book of Practical Mathematics definitely intended to satisfy the new Army Regulations. The portion given in this volume covers very well the syllabus for the "Qualifying" Examination, and its equivalent, the "Practical Measurements" of the Leaving Certificate Examination.

The first two chapters on contracted methods and graphs are scarcely necessary, as most of the work is now well done in books on arithmetic and algebra. In the graphs the writers are not uniform in indicating the scales used. It seems desirable that in *all* cases the numbers should be plainly marked on the axes. For some reason the "cooling curve" on p. 31 is inverted.

The chapters on length, mass, area, and volume are very fair. Mass and weighing are taken before areas and volumes. It is doubtful whether this is the best plan, but the reason in this case was evidently to prepare for methods of obtaining areas and volumes by use of the balance.

Some practical work might be done with simple screws before students use the screw-gauge and spherometer, and in the application of the latter instrument for finding the radius of a sphere the authors give only the method by formula. As mentioned above the alternative method by geometrical construction drawn to scale is more easy to remember, and for approximate results is quite as good.

No reasons are given for the precautions to be taken in using a balance.

The experimental work and examples are well selected. The practical work on density is especially good. The term "specific gravity" seems a favourite one with the authors. It seems a pity that the word "density" is not used more.

Trapezoid is used for the more generally accepted trapezium, and grs. for gms.

On p. 88 occur the following statements:

loss of weight = 2·96 grs. = volume in cc.

The authors have, on the whole, provided a useful preliminary course, but perhaps more attention might be given to graphical work in connection with the experiments, *e.g.* to show experimentally that pressure varies with depth.

The diagrams are good, and the text is clear. Answers to examples are given at the end.

There is a footnote on p. 60, in which it is stated that the weight of a body at the bottom of a mine is diminished owing to the upward attraction of the mass of the earth above it. This is an unusual method of simplifying a problem in Attractions.

Practical Mathematics. STERN and TOPHAM. (Bell & Sons.)

This is the complete work of which the foregoing is a part. The additional chapters commence with the Slide Rule. This is an excellent innovation for a school text book. The authors suggest very briefly the method of making a working instrument. The actual construction is a most instructive exercise. Some excellent slide rules were made by boys at Bedford Grammar School in their spare time. A use of the slide rule which at once appeals to students is the conversion from one unit to another, *e.g.* inches to cms. and *vice versâ*. This is not explicitly mentioned in the book.

In chap. xi. the authors discuss graphs from a general point of view. For a book on Practical Mathematics it would seem more desirable to encourage students to discover general laws from the practical consideration of special cases. In fact the whole chapter is out of place, as the substance of it should be found in books on theoretical algebra.

Chaps. xii.-xiv. on vectors (principally velocities and forces), and centre of gravity, are theoretically interesting, but contain the barest amount of practical work, unless the mere construction of figures to scale is worthy of that name.

Practical work on the mechanical powers is well described, but more from the point of view of what the authors have done than of practical work to be done by the student. Work on simple levers is very meagrè. The practical treatment of the wedge by apparatus of the authors' designing is, however, ingenious.

Dynamics occupies the portion of the book from chap. xvii.-xxi. A good deal of the work is theoretical (and as such, well done), but many of the experiments are beyond the range of practical work in schools.

Chap xxii. is on the Barometer and Boyle's Law. In each case the essential facts are given before any experimental work is done, the principle of the authors being apparently that the use of experiment is to verify knowledge obtained by hearsay or by reading. A training entirely according to this method seldom leads a student to original investigation, and is in many cases a bar to real progress.

Chap. xxiii. gives some interesting work (chiefly theoretical) connected with moments of inertia. Chap. xxiv. is on centres of pressure, and contains an experiment which might have been placed much earlier in the book, viz. to prove that pressure increases uniformly with depth.

The last chapter is on elasticity, torsion, etc.—subjects which more properly belong to physics.

On the whole, though the book has many good features, it is difficult to see in what way it justifies its title of "Practical Mathematics."

J. E. Boyt.

Anfangsgründe der darstellenden Geometrie für Gymnasien. By F. Schutte. (Teubner.)

A useful brochure which in the space of 41 pages explains the aim and methods of Descriptive Geometry. After a brief introduction the author proceeds to the explanation of Monge's method of representing the form of a body by means of its orthogonal projections on two planes, at right angles to each other. This occupies about half his space. His next section deals with Oblique Parallel Perspective, and the last with Central Perspective. It can be heartily recommended to those who wish to get a simple explanation of principles. We notice with interest a reference to the Stellar solids of Poinsot, one of which, the "Convexe Sterndodekaeder," is figured.

Parallelperspective—Rechtwinklige und Schiefwinklige Axonometrie. By Prof. J. Vonderlinn. (Göschen, Leipzig.) 80 pf.

An excellent work, which in 112 pages gives an account of the various methods for representing the orthogonal and oblique projections of solids, whose position is known with reference to a system of three rectangular axes. We know of no corresponding work in English. The note in the *Gazette* (177) giving the substance of De Morgan's article, "On the Projection of Mathematical Figures," will give a good idea of the nature of its contents. To any teachers who happen to be unfamiliar with this branch of mathematical study, and who wish for a simple exposition of its principles, this little book may be as heartily recommended as the preceding one. Doubtless the subject will by degrees win its way into the ordinary mathematical curriculum, especially as the Civil Service Commissioners have introduced the elements of it into their programmes of examination work.

Planimetrie für das Gymnasium. By G. HOLZMÜLLER. **Erste** Teil. Zweite Auflage. (Teubner.)

A course of Plane Geometry up to and including proportion, with practical applications. It will bear comparison with any of the numerous works of the same range that have been published in the United Kingdom since 1902. The treatment is able and suggestive, the diagrams excellent and done on a generous scale, *e.g.* those on the Radical Axis and Coaxal Circles. Like Dr. Mackay, the author has inserted many historical notes and references.

Auslese aus meiner Unterrichts- und Vorlesungs-praxis. By PROF. DR. H. SCHUBERT. Erste Band. (Göschen, Leipzig.)

An interesting work dealing with several subjects, of which the ordinary mathematical student finds it difficult to obtain the information he wants without consulting large works which are difficult of access. Of its 10 sections, II. (on the division of the circle into 17 equal parts), VIII. (on the 'Konstantenzahl' of a polyhedron), and X. (Circles and Spheres), strike us as the most attractive. The last gives among its six sub-sections Steiner's solution of the problem, *To describe a circle to touch 3 given circles.* In I. the author shows how the logarithms of the prime numbers may be calculated successively by means of the conditions:

$$-\log(x-1)+2\log x-\log(x+1)>0,$$
$$-\log(x-1)+3\log x-3\log(x+1)+\log(x+2)>0,$$

and so on, used in the same manner as equations to eliminate all but the one sought.

Plane Geometry. Books IV. and V. By J. S. MACKAY. (W. and R. Chambers.) 2s.

An excellent little treatise giving the essentials of Euclid V. and VI., and additional theorems important in modern geometry. Practical exercises are also given. A short section on Incommensurables, which follows their treatment by Prof. G. A. Gibson, affords an introduction to their difficulties and a sufficient illustration of the way in which these difficulties may be overcome. The selection of theorems, both for book-work and exercise, seems judicious. Among these we may single out for special mention the theorems of (1) Menelaus, (2) Ceva, and (3) Desargues. Some neat arrangements are given for quickly writing down the results of (1) as referring to a set of four lines, and of (2) as referring to a set of four points. Appended to the ordinary proof of (3) is an agreeable innovation in the shape of a reference to Solid Geometry. We hope that teachers will follow the course indicated and have made by their pupils the simple models described. We think the example here set by Dr. Mackay might be profitably imitated. We are convinced that the minds of many students are stunted by being kept too long and too exclusively on plane geometry, and that an occasional excursion into three dimensional territory would prove in many cases a useful stimulus. INVERSION is taken *pari passu* with SIMILARITY, as in the author's paper published in the *Proceedings of the Edinburgh Mathematical Society* (vol. vi. 1887-8). Interesting as this parallelism is, we are inclined to think that in working it out the author allows more space to Inversion than it deserves in a school book. We would rather see the pages occupied by applications of the theory of Similarity to geometrical problems. As the subject of *Plane Perspective* is introduced, the origin of some of the thumb rules of the draughtsman would have afforded excellent material in this connexion. Interesting algebraical applications are not wanting, and frequent historical notices are supplied from the store of erudition the author has accumulated in the course of his extensive investigations.

Elementary Modern Geometry. Part I. By H. G. WILLIS, M.A. (Clarendon Press.)

In this part the author treats of Triangles and Parallels, the discussion of the axioms on which the theory of the latter is founded being more thorough than is usual in school text-books. In the treatment favoured by the author, parallel straight lines are defined as "straight lines which meet at infinity," the meaning of the phrase having been explained by a consideration of "limiting positions," an alternative treatment by Playfair's axiom being supplied for those who are not yet advanced enough for the other. It is an agreeable change for the reviewer, and possibly for the teacher, to come across a lucid explanation of elliptic, parabolic and hyperbolic spaces in a school manual, but we have our doubts as to its effect on the mind of Jones tertius. Possibly his *blandi doctores* will allow a little skipping here and let him remain mentally a little longer in the parabolic world in which he has been used to dwell. Loci and problems receive adequate treatment, and the practical course contains some useful sections on graphs.

Experimental and Theoretical Geometry. By A. T. WARREN, M.A. (Clarendon Press.) 3rd edition

The fact that a third edition of this treatise has been called for seems to show that its merits (to which we drew attention in the *Gazette* of July, 1903) have secured for it a substantial share of the favour of reforming teachers A useful feature of this edition is the inclusion of various examination papers set under the new system, and occupying upwards of 30 pages. E. M. LANGLEY.

The Power of the Continuum. By DR. PITTARD-BULLOCK, Charlottenburg, Berlin, 1905. 42 pages. (Publisher's name not given.)

If we can set up a (1, 1) correspondence between the elements of two classes, we say that the classes have the same power (or cardinal number). Now, one of the most important discoveries of Georg Cantor was the proof (1873) that the class of positive integers 1, 2, 3 ... has the same power as the class of real algebraic numbers (which include the rationals) in the interval (0 ... 1), and has a less power than the continuum,—the class of all real numbers in (0 .. 1).

On the other hand, Cantor, by the construction of certain "well-ordered" series,* showed that the latter define powers which can be arranged in an unbroken ascending sequence. In particular, it results that there is a next greater power to the above power of the "enumerable" class 1, 2, 3, ...,—which latter power is seen, without difficulty, to be the least infinite power,—but it has never been proved that the power of the continuum is the least power but one, although many, including Cantor, are convinced that this is so. Dr. Pittard-Bullock's aim is to prove it.

After some historical reflexions and a repetition of some known theorems in his own terminology (*e.g.* "dinumerable" for "enumerable," "eutaxitic" for "well-ordered" †) Dr. Pittard-Bullock states the theorem (p 28),

"Every multitude of lower power than that of the *n*-dimensional continuum ($n > 0$) is dinumerable."

Of course, from this theorem would follow that the continuum has the second power, and can be well-ordered, but Dr. Pittard-Bullock does not give the proof. He refers to a yet unpublished work of his on "Transcendental Numbers in Theory of Multitudes." We may remark, however, that the grounds he gives for asserting the above theorem do not seem sufficient. In his own words (pp. 26-28), his proof that the continuum has the second power ' is partly based on a reflection that multitudes of lines, the lengths of which approach the lower limit zero (excl.), and which are found by imagining an infinite straight line divided *ad libitum*, though with the restriction, that two particles have no more than one point in common, is ‡ dinumerable. . ." Also, if A is any enumerable class, and T is the class of transcendental numbers (which is of the power of the continuum C):

"Cumulation of dinumerable submultitudes of T to A would result in a dinumerable multitude, addition of non-dimunerable submultitudes would result in a non-dinumerable multitude of at least the power of C. The third possibility, addition of a non-dinumerable multitude of dinumerable multitudes, would likewise result in a multitude equivalent to C, or of higher power, as the multitude P of points on a line, or of lines, L, the lengths of which approach the lower limit zero (excl.) are dinumerable. Assuming P to be non-dinumerable, the non-dinumerability of the first elements of the elements of P would follow, and thus the non-dinumerability of A enlarged by P."

The first statement is correct, at least if the "submultitudes" are finite in number or enumerable; the second assumes the theorem to be proved, for, if the latter be false, we can add a non-enumerable (of the second power) class to A without making it equivalent to C; the third is incomprehensible as it stands, but it may be remarked that, that the addition of a class of the second power, of enumerable aggregates, gives a class of the power of C assumes again the theorem to be proved.

---

* When we speak of a *class*, we do not assume any particular *order* of the elements. A "series" is an ordered class.

† On p. 12 we read of the "arithmetics of ordinalia" (the laws of operation with ordinal numbers).

‡ Presumably *are* is meant. The theorem is Cantor's (*Math. Ann.* xx. 1882, p. 117).

The rest of the pamphlet is taken up with the proof that (p. 31) "the multitude W of types is also of the lowest power but one." This, as stated, is false; that the class of types of *enumerable* series is of the power of C is true (Bernstein), but irrelevant here; that the class of *enumerable, well-ordered* series is of the second power (which is what Dr. Pittard-Bullock really proves) has been proved in 1882 by Cantor, but only shows that C can be well-ordered if we have proved that there is no power between that of A and that of C.

It is easier to discover the faults of a proof when regard is paid to orthography and grammar. PHILIP E. B. JOURDAIN.

Einleitung in die Funktionentheorie. VON O. STOLZ und J. A. GMEINER (II. Abtheilung, pp. 243-598, Leipzig, Teubner, 1905).

The two parts of Stolz and Gmeiner's *Theoretische Arithmetik*, and the first part of the *Einleitung in die Funktionentheorie*, have been already noticed in these columns, and it is sufficient praise of the present volume to say that it is in every way as excellent as its predecessors. The 'second edition' of Prof. Stolz's well known *Vorlesungen ueber allgemeine Arithmetik* is now complete. No part of the original work has been more thoroughly revised or more judiciously enlarged than that which furnishes the subject matter of this volume.

A new chapter has been added on the Weierstrassian notion of the analytic function. After reading it one is disposed at first to regret that the authors do not go further into the developments of the theory, but it is probable that they have gone as far as it is wise to go without invoking the aid of the differential coefficient and the integral.

It is not easy to find anything to criticise adversely. The authors cannot lay claim to the conciseness of Jordan or the sprightliness of Picard: but in lucidity, thoroughness, and consistency of purpose they yield to no one, and although the book is not exactly easy reading it is never unreasonably difficult or heavy. It is enlivened with a large number of excellently chosen examples, many of them interesting and important theorems in themselves.

If there is one chapter which seems to me better than the rest it is chapter viii., "Die Kreisfunktionen für komplexe Werte des Arguments." The sections dealing with the equation

$$e^{\xi i} = \cos \xi + i \sin \xi,$$

the binomial theorem for a complex variable and exponent, and the functional character of the logarithm, are particularly good. The excursion into the theory of the Bernoullian numbers, and their use in asymptotic approximations, is very welcome. The chapter on Infinite Products contains a number of very useful theorems concerning products whose factors are functions of a variable, which have so far only appeared in memoirs, if at all. The chapters on continued fractions, it must be confessed, are a little less attractive; but continued fractions, indispensable and interesting as they are, are particularly difficult to treat in an attractive manner. G. H. HARDY.

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THE

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THE DISCUSSION OF CERTAIN POWER-SERIES.

I QUITE agree with Mr. Hardy (*Gazette*, vol. 3, p. 284) in his contention that the real basis of the power-series for $\sin x$ and $\cos x$ is given by the two equations

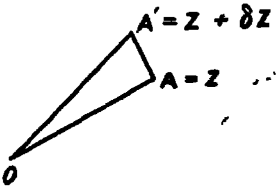
$$y=\cos x+i\sin x,\quad \frac{dy}{dx}=iy. \qquad \text{.................(1)}$$

But I am inclined to doubt whether at present the difficulties of exposition will not outweigh the advantages of his method (iii); and for this reason the following method may be worth considering, although it is a fairly obvious modification of my previous note (*Gazette*, vol. 3, p. 85). The only preliminary knowledge implied is that of the elementary differential calculus, and of Argand's diagram.

Lemma.

If Z is a complex variable depending in any way on a real positive increasing variable x, we have

$$\frac{d}{dx}|Z|\leqq\left|\frac{dZ}{dx}\right|.$$



For, in the diagram, we have

$$OA=|Z|,\ OA'=|Z+\delta Z|,\ AA'=|\delta Z|,$$

and so

$$\frac{d}{dx}|Z|=\lim_{\delta x=0}\frac{OA'-OA}{\delta x}\leqq\lim_{\delta x=0}\frac{AA'}{\delta x}\leqq\left|\frac{dZ}{dx}\right|.$$

Thus if a real positive function X, increasing with x, can be found such that $\left|\frac{dZ}{dx}\right|\leqq\frac{dX}{dx}$, we have $\frac{d}{dx}[X-|Z|]\geqq 0$, and accordingly $X-|Z|\geqq 0$, for positive values of x, provided that X and $|Z|$ are zero for $x=0$.

If we assume that y can be expanded in a power-series, and that the series can be differentiated term-by-term, there is no difficulty in seeing that y must be equal to

$$1+ix+\frac{1}{2!}(ix)^2+\frac{1}{3!}(ix)^3+\dots.$$

But to obtain a rigid proof of the equation we must proceed differently: consider the functions

$$\left.\begin{aligned}&Z_0=y,\ Z_1=y-1,\ Z_2=y-(1+ix),\\&Z_n=y-\left[1+ix+\frac{1}{2!}(ix)^2+\dots+\frac{1}{(n-1)!}(ix)^{n-1}\right]\end{aligned}\right\}. \quad \dots(2)$$

Then from (1) and (2) we see that

$$\frac{dZ_1}{dx}=iZ_0,\ \frac{dZ_2}{dx}=iZ_1,\ \dots,\ \frac{dZ_n}{dx}=iZ_{n-1}. \quad \dots\dots\dots\dots\dots(3)$$

Thus from (3) we have

$$\left|\frac{dZ_1}{dx}\right|=|Z_0|=|y|=1,$$

and therefore since $Z_1=0$ for $x=0$, we find from the lemma,

$$|Z_1|\leqq x. \quad \dots\dots\dots\dots\dots\dots(4)$$

From (3) and (4) we get

$$\left|\frac{dZ_2}{dx}\right|=|Z_1|\leqq x,$$

and hence $$|Z_2|\leqq\tfrac{1}{2}x^2. \quad \dots\dots\dots\dots\dots\dots(5)$$

Proceeding thus we find, step by step,

$$|Z_n|\leqq\frac{1}{n!}x^n. \quad \dots\dots\dots\dots\dots\dots(6)^*$$

Hence $$\lim_{n=\infty} Z_n=0,$$
and therefore

$$y=1+ix+\frac{1}{2!}(ix)^2+\dots \text{ to } \infty.$$

It is easy to prove by exactly the same method that if η is a function (real) such that $\frac{d\eta}{dx}=\eta$, then

$$\left|\eta-\left(1+x+\frac{1}{2!}x^2+\dots+\frac{1}{(n-1)!}x^{n-1}\right)\right|<\frac{1}{n!}Hx^n \quad \dots\dots(7)$$

where H is an upper limit to $|\eta|$ between 0 and x.

---

* The remainder formula (6), is new as far as I know.

Although a little different from the main object of this note, it may be useful to point out the corresponding arrangements for dealing with the binomial expansion. Let us write

$$X_n = 1 + mx + m_2x^2 + \ldots + m_{n-1}x^{n-1} \quad \ldots\ldots\ldots\ldots\ldots(8)$$

where the coefficients are of the usual form.

Thus
$$(1+x)\frac{dX_n}{dx} = mX_n - (m-n+1)m_{n-1}x^{n-1}$$
$$= mX_n - n(m_nx^{n-1}), \quad \ldots\ldots\ldots\ldots\ldots(9)$$

or*
$$\frac{d}{dx}\left[\frac{X_n}{(1+x)^m} - 1\right] = -\frac{n(m_nx^{n-1})}{(1+x)^m}. \quad \ldots\ldots\ldots\ldots\ldots(10)$$

Hence, when x is positive, (10) gives

$$\frac{d}{dx}\left|\frac{X_n}{(1+x)^m} - 1\right| \leqq n\,|m_n|\,x^{n-1}$$

so that
$$\left|\frac{X_n}{(1+x)^m} - 1\right| \leqq |m_n|\,x^n,$$

or
$$|X_n - (1+x)^m| \leqq |m_n|\,.\,x^n(1+x)^m. \quad \ldots\ldots\ldots\ldots(11_1)$$

But, if x is negative and greater than $-a$, where a is a positive proper fraction, we get from (10)

$$\left|\frac{X_n}{(1+x)^m} - 1\right| \leqq \frac{|m_n|\,a^n}{(1-a)^m},$$

or
$$|X_n - (1+x)^m| \leqq |m_n|\frac{a^n}{(1-a)^m}. \quad \ldots\ldots\ldots\ldots\ldots(11_2)$$

T. J. I'A. BROMWICH.

ON THE ADJUSTMENT OF KATER'S PENDULUM.

THE measurement of the acceleration due to gravity by means of the compound pendulum is a very common laboratory experiment; but there are certain simple principles involved in the theory of it which are not, as it seems to me, sufficiently emphasized by teachers, and which are not stated (at all events explicitly) in any text-book known to me. It may be of use to some to enunciate them here, and to shew their importance both to the instrument-maker and the student.

1. One question that sometimes arises is, What will be the effect of shifting an adjustable load to different places along the bar of a compound pendulum?

---

*I suppose that the rule for differentiating x^m has been proved *independently of the binomial theorem*; the proofs which depend on the binomial (for indices which are not positive integers) appear to me objectionable in every way, at least for beginners.

It is easy to predict the effect of *adding* a particle; but what will be the effect of moving a particle already present? The first case may be at once dismissed by saying that if a particle be added above the centre of oscillation, it tends to swing as a simple pendulum of shorter period, and therefore accelerates the pendulum; while if added below, it retards it. The second case requires further consideration. Let us suppose the particle to be moved along a straight line, called the axis of the pendulum, passing through the centre of suspension and the centre of gravity. Let the particle be of mass m, the mass of the rest of the pendulum being M; let the positions of m and the C.G. of M respectively be denoted by co-ordinates x, h, reckoned along the axis from the centre of suspension. Let the original and altered co-ordinates of m be x_1, x_2, respectively; let the moment of inertia of M about the knife-edge be I; let t_1, t_2 be the periods corresponding to x_1, x_2, and let l_1, l_2 be the corresponding simple equivalent pendulums. Then $t_2 >=< t_1$ according as $l_2 >=< l_1$, or as $\frac{I+mx_2^2}{Mh+mx_2} >=< \frac{I+mx_1^2}{Mh+mx_1}$. Now the alteration in the numerator is $m(x_2^2-x_1^2)$, and this expressed as a fraction of the original numerator is $\frac{m(x_2^2-x_1^2)}{I+mx_1^2}$; similarly the fractional increase in the denominator is $\frac{m(x_2-x_1)}{Mh+mx_1}$. The above condition amounts then to $\frac{m(x_2^2-x_1^2)}{I+mx_1^2} >=< \frac{m(x_2-x_1)}{Mh+mx_1}$, *i.e.* to $(x_2+x_1) >=< l_1$, when $x_2 > x_1$; if $x_2 < x_1$, the signs of inequality must be reversed. This condition may be put into a very simple form. Let us call the point which lies midway between the centres of suspension and oscillation for the *first* position of m the "middle point"; let us call points on the axis situated symmetrically with respect to the middle point "symmetric points." Then if a particle is initially above the middle point, raising it will retard the pendulum, lowering it to any point short of the symmetric point will accelerate it, lowering it beyond the symmetric point will retard it. If it is initially below the middle point, lowering it will increase the period, raising it to any position below the symmetric point will decrease it, raising it still further will increase it. The disadvantage of this method of expressing the results is that we utilise l_1, which is not a constant. Let us now proceed rather differently, taking L as the simple pendulum equivalent to the compound pendulum *without the particle*; thus L is constant.

We wish to discover a value x_2 which will give to the expression $\frac{I+mx^2}{Mh+mx}$ the value $\frac{I+mx_1^2}{Mh+mx_1}$. This leads to a quadratic, of which of course one root is x_1; the other is $\frac{I-Mhx_1}{Mh+mx_1}$. Putting $L=\frac{I}{Mh}$, this reduces to $x_2=\frac{L-x_1}{1+\frac{mx_1}{Mh}}$. Moving the particle within the limits x_1 and x_2 as given by this expression will accelerate the pendulum; transferring it to a point outside these limits will retard it. Moving it from the upper limit to any position short of the lower limit increases the restoring couple more than the moment of inertia; moving it beyond the lower limit increases the moment of inertia more than the couple; raising it from the upper limit to any position short of the knife-edge decreases the couple more than the moment of inertia; raising it still higher will increase the moment of inertia while still decreasing the couple. Of course there must be positions of m which will make the period a minimum; these correspond to the cases when

$$x_2=x_1=-\frac{Mh}{m}+\sqrt{\left(\frac{Mh}{m}\right)^2+\frac{Mh}{m}.L}$$

and when

$$x_2=x_1=-\frac{Mh}{m}-\sqrt{\left(\frac{Mh}{m}\right)^2+\frac{Mh}{m}.L};$$

when m is a small fraction of M, the first will be very little below the middle point of L, while the second will be far above the knife-edge and need not be considered.

2. If we suppose next that the mass m is not a particle but has an appreciable moment of inertia about its own C.G., allowance can readily be made for this. In the first case, that of addition of load, the altered period corresponds to a simple pendulum $\frac{I+i+mx^2}{Mh+mx}$, where i denotes the moment of inertia of m about its own C.G. The question is then how this expression compares with $\frac{I}{Mh}$, *i.e.* with L; and there is no difference between them provided the C.G. of m is placed where

$$x=\frac{L}{2}\pm\frac{1}{2}\sqrt{L^2-4.\frac{i}{m}}.$$

When the radius of gyration of m about its C.G. is small compared with L, these two positions are respectively just below the knife-edge and just above the centre of oscillation of the pendulum unloaded. If m be attached at any intermediate position it decreases the period; if outside these limits it has a retarding effect.

So far as transferring m from one position to another is concerned, the only change in the previous investigation is that

$I+i$ replaces I, so that L must now be taken to mean the simple pendulum equivalent to the compound one with m so fixed that its C.G. is at the knife-edge.

3. The whole of the above discussion may be represented in a convenient graphical manner as follows. Let the axis of the pendulum be taken as x-axis, and at each point erect an ordinate equal to the length of the simple pendulum equivalent to the compound one when m is fixed at the point in question. The equation of the graph so obtained is $y=\frac{I+i+mx^2}{Mh+mx}$, which is of the form $x^2-xy=$ a constant, and represents a hyperbola. One of its asymptotes is $y=x-\frac{Mh}{m}$, the other is $x=-\frac{Mh}{m}$. These lines cross the x-axis at the two points where m would have the same moment as M about the knife-edge; the latter of the two corresponds to period infinity, as is obviously right since the C.G. of the whole pendulum is at the knife-edge. If m be placed beyond the point where $x=-\frac{Mh}{m}$, we obtain the other branch of the hyperbola, of which the ordinates are negative. This of course is a sign that under the increased influence of m the pendulum is now hanging the other way up. Provided this is borne in mind, the ordinates may be plotted as if they were positive, and the second branch will then be a reflection of the first in the asymptote. The equivalent simple pendulum is a minimum for points given by

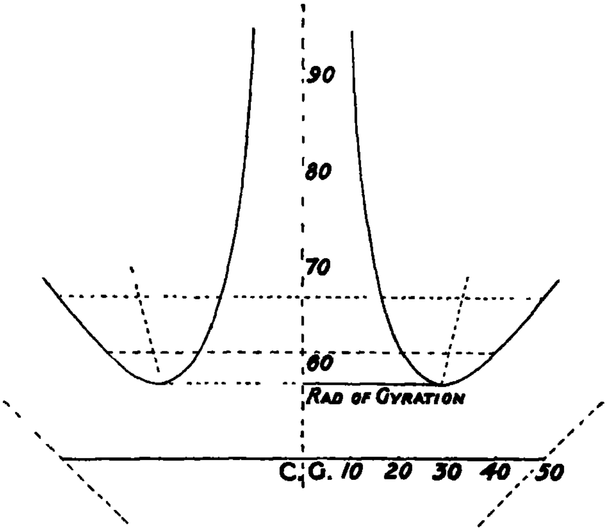
$$2x(mx+Mh)=I+i+mx^2,$$

$$\textit{i.e.}\text{ where } x=-\frac{Mh}{m}\pm\sqrt{\left(\frac{Mh}{m}\right)^2+\frac{I+i}{m}};$$

when m is in either of these positions, the corresponding simple pendulum is of length $2x$.

4. A common laboratory experiment on the compound pendulum is to swing a bar from a knife-edge whose position can be varied, and then to plot a curve shewing how the time of swing depends on where the knife-edge is, but (so far as I have seen) it is not customary to point out all that may be learned from such an experiment. Let us take x now to denote the height of the knife-edge above the C.G., which is regarded as origin; then the length of the simple equivalent pendulum is $\frac{I+Mx^2}{Mx}$, so that the graph in this case is $y=\frac{I+Mx^2}{Mx}$; the asymptotes are $x=0$ and $y=x$, of which the former corresponds to infinite period with the knife-edge at the C.G. If the knife-edge be moved still further, the bar hangs with the other end up, and we realise the other branch of the

hyperbola; the two branches may very well be plotted on the same side of the x-axis, and they will then be symmetrical about the y-axis. The minimum value of y will be twice the radius of gyration about the C.G., which is $\sqrt{\frac{I}{M}}$; and this corresponds to positions of the knife-edge at $x = \pm\sqrt{\frac{I}{M}}$. A line drawn across the graph parallel to the x-axis at any altitude exceeding $2\sqrt{\frac{I}{M}}$ will of course give four intersections, symmetrically placed with respect to the y-axis; and the distance between an asymmetric pair will be equal to the ordinate of the four points,



Uniform bar, a metre long.

since it will be the length of the simple pendulum equivalent to the compound pendulum swung from the four corresponding points. If the altitude of the line drawn exceeds $2\sqrt{\frac{I}{M}}$ only slightly, the four intersections will nearly coincide, two and two; and the length of the equivalent simple pendulum might then easily be misjudged owing to the selection of a symmetric instead of an asymmetric pair. This illustrates the point mentioned in Gray's *Text-book of Physics*, Vol. I., where the reader is cautioned against selecting two points equidistant from the C.G.; and it is stated that the outrageously wrong value obtained for g will usually reveal the nature of the mistake. It might be added that with a pendulum reasonably designed such a mistake cannot fail to be revealed; for, as will presently be seen in another way, the very essence of proper adjustment is that the period should be such as to give four well separated intersections on the graph, or (in other words) to place the C.G. at very unequal distances from the centre of suspension and the centre of oscillation. (See § 9.)

5. Another matter worth mentioning in this connection is that as the graph is a conic, the locus of parallel chords is a straight line. Therefore, when the curve representing the observations has been plotted, a series of lines parallel to the x-axis should be drawn across it; if we then confine our attention to one branch at a time, the chords should be bisected and the middle points used in order to construct a "diameter" of the conic; this will intersect the curve at a point corresponding to the shortest period. The same thing having been done with the other branch, the distance between the two minima should be ascertained; the square of half this distance multiplied by the mass of the pendulum gives its moment of inertia about its C.G.

6. The simplicity of these results is often masked by plotting as ordinate the period, instead of the length of the equivalent simple pendulum; the graph has then the equation

$$\frac{y}{2\pi} = +\sqrt{\frac{Mx^2+I}{Mgx}},$$

a cubic whose only asymptote not entirely at infinity is $x=0$. It has no points below the x-axis or on the negative side of the y-axis, and in fact when the knife-edge is taken to the other side of the C.G. the equation requires a change of sign in the term Mgx; it then gives a new curve which is the reflection in the y-axis of the previous one, and corresponds to cases in which the pendulum hangs the other way up. It will be noticed that although in general a straight line cuts a cubic in three points, in this case a line parallel to the x-axis will only cut it in two; yet imaginary points can only come in in pairs. Of course the explanation is that the third point has gone away to infinity. Had there been three intersections at a finite distance, there must have been three points on the same side of the C.G. giving the same period; whereas we know there are only two. Similarly a line parallel to the y-axis meets the curve in one point at infinity, and the y-axis itself meets it in three points at infinity. The experiment is often carried out with a uniform bar drilled with holes along its length; each of these in succession is used as a means of supporting the bar upon a knife-edge. This method is preferable to attaching a knife-edge to the bar and sliding it to different positions. With a uniform bar the cubic character of the curve is not noticeable, for the point of inflexion lies beyond the end of the bar; viz., at a distance from the centre equal to $\sqrt{1+\frac{2}{3}\sqrt{3}}$ times the half-length. However, by loading the bar at the centre the point of inflexion may be brought nearer; *e.g.* if at the C.G. of the bar there be attached a small mass having thrice the mass of the bar, then the point of inflexion is at only half the above distance.

If the experiment is carried out by means of a sliding knife-edge clamped to the bar, the only effect on the expression for the simple equivalent pendulum is to increase the numerator by one constant and the denominator by another, and the character of the graph is unchanged. A. O. ALLEN.

REVIEWS.

Volume and Surface Integrals used in Physics. By J. G. LEATHEM, M.A. (Cambridge University Press, 1905.) Price 2*s.* 6*d.* net.

This is No. 1 of a series of "Tracts" in the course of appearance dealing with mathematics and mathematical physics. So far as we can see, the object of these tracts is to deal with portions of mathematical theory too small to form a book, and either too large to form a paper or of too general application to remain buried in transactions until some specialist unearths them. If this view is correct, the useful purpose served by such tracts might almost justify their being multiplied indefinitely; indeed the "Tract" method of publication has many advantages over the "Transactions" method. The volume and surface integrals discussed in the book have special reference to quantities satisfying the partial differential equations of the first and second degrees occurring in physics. In the introduction the author discusses at some length the applicability of infinitesimal analysis to the properties of bodies whose ultimate structure is molecular, and suggests the term "physical smallness" as designating the order of magnitude of the elements involved in the analysis of physical bodies. Mr. Leathem's "physically small element" is thus the same as the "differential element" of the present reviewer. G. H. BRYAN.

Lectures on the Theory of Functions of Real Variables. By JAMES PIERPONT, Professor of Mathematics in Yale University. Volume I., xii + 560 pages. (Ginn & Co., Boston, [1905].)

In its historical development, the theory of functions of real variables is not always separable from the theory of analytic functions of complex variables. For example, Weierstrass' continuous function with no derivative was discovered by considering the real part of a certain analytic function on its circle of convergence, which is also a line of essential singularities; further, it casts, I think, some light on the theory of functions of a real variable to reflect that the first systematic treatment of this theory was made (by Hankel in 1870) long after the theory of analytic functions, as given by Cauchy and Riemann, had come to be widely known, and that it was made with the express object of proving* that the limitation of the function-concept implied in the term "analytic" was a wise and, indeed, a necessary one. Lastly, that revision of fundamental principles, including the purely arithmetical introduction of irrational numbers, originated in the need for proofs of some vital theorems in the theory of analytic functions. To convince ourselves of this, we need only remember Weierstrass' avowed object that (at any rate, at first) he regarded the study of Abelian functions as his chief end in mathematics, and then glance at the pre-Weierstrassian "Théorie des fonctions doublement périodiques" of Briot and Bouquet, where, in No. 35, we can see the great importance for the theory of functions of Weierstrass' theorem that an infinite aggregate of points has at least one point of condensation, and, from No. 37, the importance of his distinction between upper limit and maximum, and his proof that a continuous function attains its maximum.†

While, then, Weierstrass' theory ‡ of irrationals, and its more amenable child, which is known as Cantor's theory, presupposes some acquaintance with the theory of analytic functions in order that we may fully realise their importance at the outset, Dedekind's theory, which arose from his attempts to deal with the proof of the existence of a limit in a purely arithmetical manner, would appear to be more suitable for beginners, and this view has found support in works of Pasch and J. Tannery. I am here doubting the advisability of any mode of exposition which leads to the remark of Prof. Pierpont (pp. 4-5): "It is too early to make the reader see the necessity of this step. . . ." If it is ever advisable to teach mathematics in this way, it is by introducing the student by a path which can never be used for the discovery of any important truths. It is possible, no doubt, to support such methods by the argument of the necessity of having a firm logical basis. But such arguments are never consistently applied. We do not abolish Euclid because of Mr. Russell's just strictures on its logic, and we are right; subtler logic is a result, not a prerequisite, of education. And I cannot but think that it is partly owing to unhistorical teaching that one so frequently sees the pitiable spectacle of a young man who has taken a high degree, and is therefore urged to undertake some mathematical research for a prize or a fellowship, not knowing what to do or how to set about finding anything to do.

Historically, the exact analysis of the number-concept comes late, and this analysis is perhaps the most difficult branch of mathematics. If an arithmetical theory of irrationals *is* to be put at the beginning of a book on function-theory, Dedekind's is preferable. These, in substance, are the criticisms which Prof. Pierpont's work suggests; and the criticisms apply only to the first few chapters of this otherwise admirable volume.

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* It succeeded in making this thesis plausible.

† Pages 46 and 48 of Fischer's German translation (1862) of Briot and Bouquet. Cf. pp. 47-48, 49 for a slight description of the ("essential," in Weierstrassian terms) singularity of $e^{\frac{1}{z}}$ at $z=0$.

‡ The same probably applies to Méray's theory, since Méray was, like Weierstrass, occupied with *analytic* functions.

A few remarks will show how easily mistakes creep into treatments of the number-concept. Prof. Pierpont postulates the theory of the positive integers 1, 2, . . ., and (p. 5) defines a fraction as a pair of integers (a, b), which has certain properties. He then (p. 6) identifies $(a, 1)$ with a. No integer can, under any circumstances, be identified with a pair of integers. In the same way '*positive* fractions' (pp. 12 *seq.*) are a different, though analogous, system to the fractions previously defined.

Rational numbers are classes of integers, as Prof. Pierpont says, and to most of us irrationals are classes of rationals (a definition which accords with, *e.g.*, Weierstrass', Lüroth's, and Russell's). But Prof. Pierpont merely says (p. 36): "Every regular sequence *defines* a number," but does not tell us *how*. He does not follow Heine's lead, and say it is a 'sign' (a proceeding which would make of arithmetic a trivial game), though a 'variable' is a 'symbol,' according to him (p. 118). When we speak of a 'variable,' we only mean a statement about *any* or *some* member of a certain class (of real or complex numbers usually).

From this point on all is excellent; the book is elementary, and is intended for those with a somewhat extensive, but uncritical knowledge of the calculus. It contains a rigorous treatment of continuity, differentiation, implicit functions, indeterminate forms, extremes, and single and multiple integrals (proper and improper). It does not deal with such advanced subjects as the condensation of singularities or continuous functions with no differential quotient anywhere.

Its chief excellencies seem to be: (1) The introduction of various current inaccurate statements as examples in criticism (pp. 231, 238, 240, 247, 251, 308, 321, 327, 371, etc.);* (2) analytical representations of some apparently 'lawless' functions. Thus

$$F(x) = f(x) + (g(x) - f(x)) \lim_{n=\infty} sgn(\sin^2 n!\, \pi x)$$

is $f(x)$ at rational, $g(x)$ at irrational points (pp. 202-205); (3) examples of continuous functions lacking differential quotients at isolated points, —functions often ignored by teachers (pp. 225-229); (4) the careful distinction between "any" and "every," which is often neglected.†

As a further instance of care, take Rolle's theorem (p. 246). It is necessary and sufficient that $f(x)$ should be continuous for the closed interval $a \leqq x \leqq b$, and $f'(x)$ should exist for every x such that $a < x < b$. If the aggregate for which $f(x)$ is continuous is not closed, the proof that $f(x)$ attains its extremes fails; and, on the other hand, the theorem holds even if there is (*e.g.*) no right-derivative at $x = a$. These facts are neglected in even such a book as Harnack's, which is usually considered by English students as almost unnecessarily rigorous.

The chapters on integration seem rather more advanced. On p. 356 (cf. p. 530) we have Jordan's conception of an integral defined over any measurable aggregate, while we have no mention (even afterwards) of say, continuous functions defined for any perfect aggregate, or of the four derivatives of a function.

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* Cauchy's definition of an irrational as a limit of rationals might be criticised when dealing with irrationals and the existence of limits.

Another suggestion I should like to make is that the author should illustrate that an always finite function need not be limited.

† Cf. Russell's *Principles of Mathematics*, Vol. I., 1903, p. 327 note. 'Every' is often used for 'any' in mathematical books of all nationalities.

The work is beautifully printed, and the only misprint I have seen is on p. 247, l. 15, where $f'(x)$ is put for $f(x)$.*

PHILIP E. B. JOURDAIN.

The Integration of Functions of a Single Variable. By G. H. HARDY. (No. 2 of the Cambridge Tracts on Mathematics and Mathematical Physics.)

This tract contains a number of striking and important theorems on indefinite integrals, which are due to Liouville very largely; as an example we may note (see p. 35) that $\log x = \int dx/x$ is *proved* not to be an algebraical function of x, a point which is not established in any of the ordinary books. Indications are given (p. 48) of the proof that the logarithm-integral $\int dx/\log x$, and the sine- and cosine-integrals $\int (\sin x/x)dx$, $\int (\cos x/x)dx$ are really new transcendents, not expressible in finite terms by any elementary functions.

As an example we may note that (p. 15) the integral

$$\int \frac{(ax^2+2\beta x+\gamma)dx}{(ax^2+2bx+c)^2}$$

is a rational function of x only if $a\gamma + c\alpha - 2\beta b = 0$, and is then equal to $-(ax+\beta)/a(ax^2+2bx+c)$. And again (p. 33) that

$$\int \frac{dx}{(x-p)(ax^2+2bx+c)^{\frac{1}{2}}}$$

can only be algebraical if $ap^2+2bp+c=0$.

Nearly half the tract is devoted to the integrals of algebraical functions, with reference to which the fundamental theorem is due to Laplace: "*l'intégrale d'une fonction différentielle ne peut contenir d'autres quantités radicaux que celles qui entrent dans cette fonction.*" It is also explained (p. 36) that exponentials cannot occur in such integrals; and logarithms only in the form $A\log[\phi(x)]$, where A is a constant and $\phi(x)$ is an algebraical function containing only such irrationals as are present in the integrand. But as a contrast to the theory it may be noted that the method (Greenhill's) given on p. 24 for the integration of such an integral as

$$\int \frac{(x+2)dx}{X_1\sqrt{X}}, \quad X_1 = (x+1)^2 + 2(x+2)^2, \quad X = (x+1)^2 + (x+2)^2$$

would apparently express the integral in a form containing $\sqrt{X_1}$ as well as $\sqrt{X}$. However, by writing $\xi = (x+1)/\sqrt{X}$, it is not difficult to see that the integral is really a multiple of

$$\log\left[\frac{\sqrt{(2X)}-(x+1)}{\sqrt{(2X)}+(x+1)}\right].$$

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*On p. 366. l. 20 (cf. pp. 206, 216), it would be better to put *lower limit* for *minimum* (which generally means an attained lower limit).

It seems a pity that space could not be spared to give an outline of the practical methods for integrals such as

$$\int\frac{dx}{X_1{}^3X^{\frac{1}{2}}},\quad X=x^2-x+1,\quad X_1=x^2+x+1.$$

The ordinary description given in English books expects $(x^2+x+1)^{-3}$ to be first resolved into partial fractions, and then a formula of reduction has to be applied. Now a formula of reduction is extremely useful in certain special *definite* integrals, but is very inconvenient for *indefinite* integration. Undoubtedly the best method is to assume

$$\frac{1}{X_1{}^3X^{\frac{1}{2}}}=\frac{d}{dx}[(Ax^3+Bx^2+Cx+D)X^{\frac{1}{2}}X_1{}^{-2}]+(Ex+F)X^{-\frac{1}{2}}X_1{}^{-1},$$

and find $A, \ldots, F$, so as to make this an identity, since the calculation involves little more labour than the determination of the partial fractions, and no reduction formula is required. T. J. I'A. B

Integralrechnung. By W. FRANZ MEYER. (No. 11 of the *Sammlung Schubert.*) (Leipzig, G. J. Göschen.)

In many respects this book differs from the customary texts on Integral Calculus: as an example, the systematic discussion of types of integrals does not appear until p. 280, and a large number of special areas, volumes, etc., have been evaluated previously by comparatively elementary methods.

Some care is taken in the discussion of the formulae for the arc of a curve and for the area of a surface of revolution (§§ 11 and 13); in this respect it is pleasant to see an improvement on the crude methods adopted in most English books. In discussing the volume of an ellipsoidal or hyperboloidal slice, however (§ 10), an application of the prismoidal formulae,*

$$\tfrac{1}{6}h(A+B+4M),$$

would have been simpler.

An interesting innovation in a Calculus is the large number of "Umkehrungen," in which curves are determined by means of some geometrical property; for example (§ 20) the curves in which (i) ρ/n, (ii) ρ/n^3, (iii) ρn are constant, where ρ is the radius of curvature and n is the normal. But it seems strange to an English reader that the formulae for the radius of curvature and allied results are established in an Integral (instead of a Differential) Calculus; and that some 20 pages are devoted to the curvature of twisted curves and surfaces.

In §§ 35, 36 are given some theorems on the elliptic integrals and elliptic functions; the addition-theorem is obtained from the integrals (using Darboux's method) in the form:

If $\quad u=\int_0^x X^{-1}dx,$ and $x=f(u)$, where $X^2=ax^4+2bx^2+c$,

then $\quad f(u+v)=c^{\frac{1}{2}}[f(u)f'(v)+f(v)f'(u)]/[c-a\{f(u)f(v)\}^2].$

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* Here h is the thickness, A and B the areas of the two parallel end faces, M the area of the central section parallel to the faces. The formula applies to any solid in which the area of the section is a quadratic (or cubic) function of x, the distance of the section from an end. For an ellipsoid the result is $\frac{1}{2}\Delta h+\frac{1}{6}A_0h^3/p^2$, where A_0 is the area of the parallel central section and p is the perpendicular on the parallel tangent-plane.

This, of course, includes the addition-theorems for the hyperbolic and trigonometrical functions as special cases.

In § 38 some of the familiar power series are derived by means of the Integral Calculus; and this is a very good method for many of the series. Thus the series for arc sin x is obtained far more easily from $\int_0^x (1-t^2)^{-\frac{1}{2}}dt$ than by applying Taylor's theorem. The book closes with a section on double integrals and their applications.

It can hardly be doubted that most teachers of the Calculus will find many points of interest and means of improving their teaching in this book. T. J. I'A. B.

Theorie der ebenen algebraischen Kurven höherer Ordnung. By HEINRICH WIELEITNER. (No. 43 of the *Sammlung Schubert.*) (Leipzig, G. J. Göschen.)

This text-book covers to some extent the same ground as Salmon's *Higher Plane Curves*, the harder portions being omitted. But several points are considered more fully than by Salmon.

Perhaps the most striking contrast between the two books may be found in the profusion of diagrams given by Wieleitner; nearly 50 of these have been drawn accurately to scale, from the corresponding equations. In particular, the diagram of the parallel curve to an ellipse (p. 288, fig. 73) is extremely instructive, quite apart from its beautiful drawing.

Amongst other matters not given by Salmon we may refer to the constant use of Newton's diagram for the discussion of singularities on a curve; to the resolution of higher singularities into nodes, cusps, inflexions and bitangents; and to Klein's theorem on *real* singularities. Frequent use of Plücker's "Auflösung" of a node is made; thus (p. 92) from the line and circle $y(x^2+y^2-1)=0$, we can get various families of cubics, such as

$$y(x^2+y^2-1)+\lambda=0, \quad y(x^2+y^2-1)+\lambda x=0, \quad y(x^2+y^2-1)+\lambda x^2=0.$$

Further, when λ is *small*, these cubics differ but little from the line and circle, which enables us to realize their shape very easily.

The book seems to be a very useful introduction to the theory of Higher Plane Curves. T. J. I'A. B.

Mehrdimensionale Geometrie. (II. Teil.) By P. H. SCHOUTE. (No. 36 of the *Sammlung Schubert.*) (Leipzig, G. J. Goschen.)

According to the preface, the author was forced, for lack of space, to compress his MS. into about three-fourths of the original number of pages; and this accounts for the occasionally rather too compact form of the argument.

The subject-matter corresponds roughly to the work given, for ordinary space, in such books as Nixon's *Geometry in Space.* The book contains extensions of Euler's theorem, of the theory of regular polyhedra, of spherical geometry, and so on; the work is well illustrated by numerous carefully drawn figures. We may mention specially the figures, for ordinary space, of the regular and semi-regular solids.

T. J. I'A. B.

The Theory of Determinants in the Historical Order of their Development. Second edition. By T. MUIR, LL.D., F.R.S. 1906. (Macmillan.)

The quality of the research undertaken by Dr. Muir in a subject he has made peculiarly his own, is too well known to every mathematician who is interested in the historical development of the theory of determinants to render it necessary for us to do more than to call the attention of our readers to the completed volume which now lies before us. *Part I.* was published sixteen years ago, and no doubt the accomplished author looked forward to the appearance of Part II. within a reasonable period after that of its predecessor. But the proposals of man are often disposed of in an unexpected manner, and the pressure of official work in a new sphere of interests and utility proved the truth of the adage. As time permitted, the lists which appeared in the *Quarterly Journal* were added to, and from the titles of the various papers collated by the author, 1740 in number, we see he has brought his labours up to the year 1900. The present volume is said to be a second edition. This is true only of the first section, covering 288 pages. Such intercalations are here made as were found necessary from discoveries made since the first volume appeared. The total amount of matter added to what was to be found in the old edition is some fifteen pages. The remaining 200 pages are what would have been Vol. II., but it was considered advisable to place the whole within one moderately-sized volume, and this is what has been done. Let us say at once that in one respect the new volume is easier to read than its predecessor—it is in larger type, and where the reader who has passed the bounds of middle age can read large type he prefers to do so as a rule. Referring to the pages of the original volume we may note a few of the additions and alterations. By a strange slip in the first edition the memoir by Garnier (1814) on p. 215, was out of its chronological place. This has been corrected, and it is now on pp. 135-6. A memoir by Cauchy (1829), dealing with the equation by means of which the secular irregularities of the movements of the planets are determined, was overlooked. Strictly speaking, the determinants in this memoir are axisymmetric, and would, therefore, fall into Part II., but as the proof given for one of the theorems in the memoir is applicable to determinants of every type, it is now included here, and placed between the memoirs of Reiss and Jacobi on p. 184 of the first edition. The one-page summary of Jacobi's memoir, just mentioned, is now replaced by five-and-a-half pages. The summary is much more detailed, and includes a proposition in which is given the generating function of the reciprocal of a determinant. At the old page, 201, after the Latin quotation, we find three pages inserted, giving Jacobi's method of proving a certain pair of theorems. This is followed by a statement of a theorem arrived at in 1834 by Jacobi, which is also new. A short summary of a paper by Molins, 1839, is introduced just before the analysis of Sylvester's article of 1840 in the *Phil. Mag.*, vol. xvi. In the note to p. 237, first edition, "a term of the cyclic-species" is altered to "a term whose index-cycle scheme." On p. 259 there is a small addition. On p. 288 appears a page of tables similar to that opposite p. 132. This rams home the facts of the second period as has already been done for the first. We now reach the special determinants which make up the subject matter of the new Part II. The various chapters deal with axisymmetric and skew determinants, the miscellaneous determinants associated with the names of Wronski, Scherk, Schweins, Jacobi, and Sylvester, and with Alternants, Jacobians, and Orthogonants. A short "Retrospect on Special Forms," with indexes, brings to a close a piece of work of which English mathematicians need not be ashamed.

An Elementary Treatise on Pure Geometry, with numerous examples; by J. W. RUSSELL. New and Revised Edition, 1905. (Clarendon Press.)

The only fault that can be found with Mr. Russell's admirable Geometry is that it is far too full for ordinary school purposes. As a book of reference it is invaluable, for all the properties that a teacher or student will ever be likely to require are here to be found. In the new edition we find a model at the end of the book which will prove of much value in the study of the chapter on projection to those whose intuitive faculty is not highly developed. The questions have been re-arranged, those that were redundant have disappeared, all are now in larger type, and the problems are relegated to the ends of the chapters. Many a weary hunt will be saved to future readers by the adequate index. The author tells us in his preface that the method of one to one correspondence has been introduced wherever possible, but only as an alternative method. He thinks it, "though elegant and powerful, a dangerous method, on account of the difficulty of seeing whether a given construction is rational." In the proofs of the theorems more use has been made of projection, and correlative theorems have been proved by reciprocation. While we admit that all the changes in the new and revised edition have been for the better, we feel sure that we are echoing the feelings of many teachers when we regret that the time thus spent was not rather devoted to the production of a text-book which can be used in schools. For such a book, written by one who has such a mastery of the subject, we are convinced there would be a large demand.

MATHEMATICAL NOTES.

191. [V. a.]. The "appeal" in the last number of the *Gazette* has encouraged me to send a miscellaneous collection of ideas, some suggested by the contents of the number, some of slower growth.

(1) *Higher Trigonometry.* It is encouraging to have Mr. G. H. Hardy's sanction for some heretical notions that many schoolmasters have for some time entertained secretly. As to the proof of $\cos x+i\sin x=\exp(ix)$ I have used, to my own satisfaction, the proof Mr. Hardy recommends: I have been in the habit of giving it to ordinary scholarship candidates, and they did not find it difficult. I should be glad to know if Mr. Hardy would pass this form (which is borrowed from the official program of L'École des Arts et de la Commerce).

Let $$\cos x+i\sin x=\exp(y). \quad \text{......(i)}$$

Differentiating, $$-\sin x+i\cos x=\exp(y)\frac{dy}{dx}; \quad \text{......(ii)}$$

$$\therefore \frac{dy}{dx}=\frac{-\sin x+i\cos x}{\cos x+i\sin x}=i; \quad \text{......(iii)}$$

$$\therefore y=ix+c; \quad \text{......(iv)}$$

$$\therefore \cos x+i\sin x=\exp(ix+c)$$
$$=\exp(ix).\exp c. \quad \text{......(v)}$$

Put $$x=0;\text{ then }1=\exp c;$$

$$\therefore \cos x+i\sin x=\exp(ix). \quad \text{......(vi)}$$

(i) assumes the existence of a root to a transcendental equation for y; (ii) arises from the term by term differentiation of a series. Am I right in supposing that it is unnecessary to enter upon a discussion of contour integration before taking step (iv)?

(2) *Contracted Multiplication and Division.* This used to be one of the main planks of the mathematical reformer's platform. I suppose that the importance attached to these rules was a sign of the reaction against the unpractical, vulgar-fractions-and-farthings type of arithmetic. Certainly, for my own part, I used to swear by these rules; I was brought up on them, and for several years taught them with enthusiasm. They were hard work, for both teacher and taught: but both were kept going by the thought of the vast saving in labour to be effected by the use of the rules. It is true that I never actually *used* them myself, as I preferred logarithms; and I never managed to make any boy use them except to win marks. Before a boy could use the rules, he had to revise them in class for a week or so; and the better boys said that to use the rules was such a strain upon their minds that they would rather work without contracting if they had no log

books. However, we pegged away. One day I wrote to several engineers to ask their opinion as to the value of the rules in real life. One said that he used the slide-rule himself; another said that, to answer my questions, he had enquired what the rules were, and he thought they must be capital things. However, on the whole I came to the conclusion that they were but toys; that it was much easier to teach logarithms straight away; and that a great quantity of the best teaching power is being squandered in teaching rules which no one would ever think of applying except under penalties. Accordingly, I cut them out of the curriculum for which I was responsible: but made careful provision for teaching the boys in other ways such notions as significant figures, degree of approximation, etc.

I think that this is a topic which might well be discussed in your pages.

(3) *Division.* I wonder if other teachers have noticed the extreme helplessness of boys of thirteen in dealing with a division sum. The difficulty is to know which number is to be the divisor. There is, of course, a preliminary difficulty—whether to divide or subtract. "If 40 sacks of coal weigh 2240 lbs., what does 1 sack weigh?" It seems to be entirely a matter of chance whether the answer is given as $\frac{2240}{40}$ or $\frac{40}{2240}$ lbs. The only principle grasped firmly is—divide by the smaller number, for choice.

In saying that it is a matter of chance "which way" the boy divides, I am not going far enough. I seriously believe that he is more likely to divide the wrong way; this belief is based on recent experience.

This may seem a small matter, but it tends to vitiate the whole of a boy's arithmetic. In searching for the causes of this curious difficulty, I find that the following suggest themselves:—

(A) Division is, and must be, taught as a mechanical process. But is it also taught, as it should be, in connection with the meaning? In other words, a boy must know how to work out the quotient to $516 \div 3$; and he always does know this. But I am beginning to think that he often forms no concrete idea of the kind of case to which division is applicable.

The remedy for this is practical work at the earliest stages—kindergarten work; cutting up paper into so many parts, dividing shot into so many parcels, etc. (partition); and also the other kind of division—how many sets of 3 counters in a heap of 60, how many times will the yard-measure fit into the length of the room and how much is left (quotition)?

(B) The other cause of the difficulty, to my mind, is that vile phrase "divided into"—"twelve divided into three" (meaning $3 \div 12$). It will be noticed that by adding the word "parts" we reverse the operation "twelve divided into three parts" gives $12 \div 3$. Is it surprising that a boy accustomed to "divide into" is a bit foggy as to which way the division is to go in a concrete case?

CHARLES GODFREY.

192. [A. 1. b.] *Note on the Power Inequality.*

The inequality in question, which is the fundamental one in *Chrystal*, may be stated and proved as follows:

If x and y be positive and $x > y$, and m be any rational number not equal to 1 or 0, then, $mx^{m-1} \gtrless \frac{x^m - y^m}{x-y} \gtrless my^{m-1}$, *according as* $m(m-1) \gtrless 0$.

Proof.—From the identity $\frac{x^n - y^n}{x-y} = x^{n-1} + x^{n-2}y + \ldots + y^{n-1}$, we get at once,

$nx^{n-1} > \frac{x^n - y^n}{x-y} > ny^{n-1}$, n being any integer >1. ....(1)

* The above inequality can be written, as is readily verified,

$$\frac{n}{n-1}x > \frac{x^n - y^n}{x^{n-1} - y^{n-1}} > \frac{n}{n-1}y.$$

Multiplying a number of such inequalities together we get, if p and q be positive integers, $p > q$,

$$\frac{p}{q}x^{p-q} > \frac{x^p - y^p}{x^q - y^q} > \frac{p}{q}y^{p-q}. \text{ ....}(a)$$

Whence, writing $x^{\frac{1}{q}}$, $y^{\frac{1}{q}}$ for x, y and m for p/q,

$$mx^{m-1} > \frac{x^m - y^m}{x - y} > my^{m-1}, \text{ if } m > 1. \text{ ....}(2)$$

From (a), considering the reciprocals of the terms, we get,

$$\frac{q}{p}x^{q-p} < \frac{x^q - y^q}{x^p - y^p} < \frac{q}{p}y^{q-p}. \text{ ....}(b)$$

Whence, writing $x^{\frac{1}{p}}$, $y^{\frac{1}{p}}$ for x, y and m for q/p,

$$mx^{m-1} < \frac{x^m - y^m}{x - y} < my^{m-1}, \text{ if } 1 > m > 0. \text{ ....}(3)$$

Again from the identity

$$\frac{\left(\frac{1}{x}\right)^n - \left(\frac{1}{y}\right)^n}{x - y} = -\left(\frac{1}{x^n y} + \frac{1}{x^{n-1}y^2} + \dots + \frac{1}{xy^n}\right),$$

by considering the effect of increasing y to x, and then the effect of decreasing x to y, on the right-hand side, we get,

$$-nx^{-n-1} > \frac{x^{-n} - y^{-n}}{x - y} > -ny^{-n-1}. \text{ ....}(4)$$

Similarly from the identity

$$\frac{\left(\frac{1}{x}\right)^p - \left(\frac{1}{y}\right)^p}{x^q - y^q} = -\frac{\frac{1}{x^p y} + \frac{1}{x^{p-1}y^2} + \dots + \frac{1}{xy^p}}{x^{q-1} + x^{q-2}y + \dots + y^{q-1}},$$

we get,

$$-\frac{p}{q}x^{-p-q} > \frac{x^{-p} - y^{-p}}{x^q - y^q} > -\frac{p}{q}y^{-p-q}. \text{ ....}(c)$$

Whence, writing $x^{\frac{1}{q}}$, $y^{\frac{1}{q}}$ for x, y, and m for $-p/q$, we get,

$$mx^{m-1} > \frac{x^m - y^m}{x - y} > my^{m-1}, \ m \text{ being any negative quantity. ....}(5)$$

Results (1), (2), (3), (4) and (5) prove the theorem.

Gooty, India, 8.4.06. V. RAMASWAMI AIYAR.

193. [C. 2. h.] The following Exercise has the advantage of obtaining two well-known integrals simultaneously:

$$I = \int_0^\infty \frac{x^{2m}\,dx}{1 + x^{2n}}, \quad m < n,$$

$$I' = \int_0^\infty \frac{x^{2m}\,dx}{1 - x^{2n}}, \quad m < n.$$

Let ODE be a sector of infinite radius R and angle $\frac{\pi}{2n}$; then the function $\frac{x^{2m}}{1 - x^{2n}}$ has one pole $P(x = 1)$ on OD and no pole inside sector.

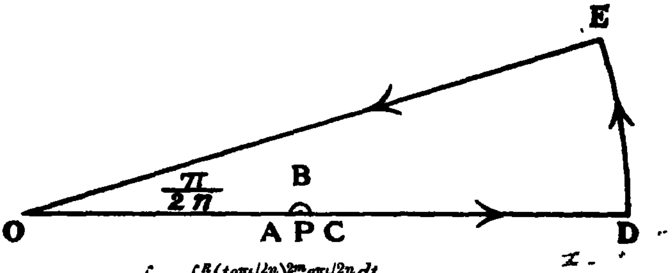
Let ABC be a semi-circle centre P and infinitely small radius r.

Integrating $\frac{x^{2m}}{1 - x^{2n}}$ over the contour excluding P, we have

$$\int_{OA} + \int_{ABC} + \int_{CD} + \int_{DE} + \int_{EO} = 0.$$

Now the 1st and 3rd make up (for $r=0$) the "principal value" of I'. Since $\underset{x=\infty}{\mathrm{Lt}}\, x \cdot \frac{x^{2m}}{x^{2n}}$ is 0, the 4th term $=0$.

Hence $$I' - \int_{OE} = \int_{OBA}.$$



Now $$\int_{OE} = \int_0^R \frac{(te^{\pi\iota/2n})^{2m} e^{\pi\iota/2n}\, dt}{1+t^{2n}} \quad \because \ e^{\pi\iota} = -1$$

$$= e^{(2m+1)\pi\iota/2n} I,$$

and $$\int_{OBA} = \underset{r=0}{\mathrm{Lt}} \int_0^\pi \frac{(1+re^{\iota\phi})^{2m} re^{\iota\phi}\iota\, d\phi}{1-(1+re^{\iota\phi})^{2n}} = \int_0^\pi \frac{\iota\, d\phi}{-2n} = -\frac{\pi\iota}{2n};$$

$$\therefore \ I' - \{\cos(2m+1)\pi/2n + \iota \sin(2m+1)\pi/2n\} I = -\frac{\pi\iota}{2n};$$

$$\therefore \ I = \frac{\pi}{2n} \Big/ \sin(2m+1)\pi/2n,$$

$$\text{and } I' = \cos(2m+1)\pi/2n \,.\, I$$

$$= \frac{\pi}{2n} \cot(2m+1)\pi/2n.$$

R. W. GENESE.

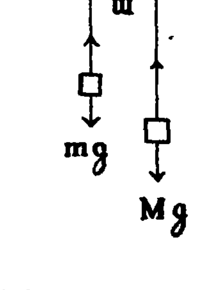
194. [R. 9; K. 9. b.] 1. *Inexactitudes in the treatment of simple exercises on the motion of unequal masses suspended over a pulley.*

(*a*) Some text-books fail to indicate clearly the idealisations involved in supposing the tension of the string to be the same throughout.

(*b*) When these several points are made clear, the reader—presumably a beginner—is left to imagine that, if two unequal masses could be suspended over a perfectly smooth pulley, this pulley would rotate.

It seems unfortunate that the part played by friction is not more clearly brought out. The ideal problem could be treated by supposing the string to pass over a smooth *peg* and by pointing out the fact that a close approximation is thus obtained to the actual motion which takes place when it passes round a suitably mounted pulley. The diagram for the "pulley" problem would then be somewhat like the accompanying figure.

[t negligible in considering the motion of M, m.]



2. *Construction of a regular pentagon.*

(*a*) The method based on Euclid IV. 10 is essentially unpractical.

(*b*) **Simple constructions, easy to perform with accuracy, are given in Plant's *Geometrical Drawing*.**

(*c*) If the latter are adopted, as seems advisable, what line of argument furnishes the easiest and most natural proofs?

Tentative answer.

(i) Consider a right-angled triangle ABC, right-angled at C, such that $\frac{BC}{CA}=\frac{\sqrt{5}-1}{4}$. Using the '$2A$' formulæ, prove that $\cos 4A=\sin A$, thus obtaining $\sin 18°$.

(ii) Employ this result in demonstrating the correctness of the constructions referred to. A. F. Van der Heyden.

195. [X 1.] *Note on the number of feet in a metre.*

A high order of accuracy is secured in calculations by taking 1 metre $=39{\cdot}37079$ inches, this being the actual value of a metre to five places of decimals. The object of this note is to show that to this order of accuracy the very simple fraction $\frac{292}{89}$ of a foot may be taken as the equivalent of a metre.

The proof is simple. Expand $39{\cdot}37079$ into a continued fraction The seventh convergent is $\frac{3504}{89}$ and the eighth quotient is 35; thus $\frac{3504}{89}$ differs from $39{\cdot}37079$ by less than $\frac{1}{35}\left(\frac{1}{89}\right)^2$, *i.e.* by less than ${\cdot}0000037$. Thus to five places of decimals they are identical, and to this order $\frac{3504}{89}$ inches, *i.e.* $\frac{292}{89}$ feet, represents a metre. H. L. Trachtenberg.

196. [L_1. 3. a.] *Cartesian Coordinates—A new method of finding the equation of the axis of a parabola.*

Let the parabola be $ax^2+2hxy+by^2+2gx+2fy+c=0$. .... (1)

Then the equation of its axis is obtained by equating to zero the linear portion of the equation,

$$\frac{(ax+hy+g)^2-(hx+by+f)^2}{a-b}-\frac{(ax+hy+g)(hx+by+f)}{h}=0. \quad \text{........(2)}$$

For had (1) been any central conic, (2) would have been the equation of its axes. But as the centre of a conic goes towards infinity, one of these axes tends to become the line at infinity. Thus in the case of a parabola one of the factors of (2) is a constant, *i.e.* the equation (2) reduces to a constant multiplied by the factor representing the finite axis. Thus the rule given above for obtaining the equation of the axis of any parabola is true. Since the highest terms of (2) have to vanish in the case of a parabola, we equate them to zero and get the condition that (1) should be a parabola. The vanishing of the three coefficients of x^2, xy, and y^2, give the same condition $h^2=ab$. H. L. Trachtenberg.

197. [L_1. 3. a.] *Trilinear Coordinates. The equation of the axis of the general parabola.*

The equation of the axes of any central conic

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$$

is

$$\begin{vmatrix} 2bc(w'\alpha'+v\beta'+u'\gamma)(v'\alpha+u'\beta+w\gamma)-c^2(w'\alpha+v\beta+u'\gamma)^2 & & \\ \qquad -b^2(v'\alpha+u'\beta+w\gamma)^2, & 2bcu'-c^2v-b^2w, & 1 \\ 2ca(v'\alpha+u'\beta+w\gamma)(u\alpha+w'\beta+v'\gamma)-a^2(v'\alpha+u'\beta+w\gamma)^2 & & \\ \qquad -c^2(u\alpha+w'\beta+v'\gamma)^2, & 2cav'-a^2w-c^2u, & 1 \\ 2ab(u\alpha+w'\beta+v'\gamma)(w'\alpha+v\beta+u'\gamma)-b^2(u\alpha+w'\beta+v'\gamma)^2 & & \\ \qquad -a^2(w'\alpha+v\beta+u'\gamma)^2, & 2abw'-b^2u-a^2v, & 1 \end{vmatrix}=0.$$

As the conic tends to become a parabola, one of the axis tends to become the line at infinity. Thus if the first equation be that of a parabola, the determinant $\equiv (a\alpha+b\beta+c\gamma)(l\alpha+m\beta+n\gamma)$, where $l\alpha+m\beta+n\gamma=0$ is the equation of the axis. Thus la = the coefficient of α^2 in the determinant, and l is found. Similarly m and n are found. Thus the equation of the axis is

$$\Sigma\left\{[(2bcw'v'-c^2w'-b^2v')\{(2cav'-a^2w-c^2u)-(2abw'-b^2u-a^2v)\}\right.$$
$$+(2cav'u-a^2v'^2-c^2u^2)\{(2abw'-b^2u-a^2v)-(2bcu'-c^2v-b^2w)\}$$
$$\left.+(2abuw'-b^2u^2-a^2w'^2)\{(2bcu'-c^2v-b^2w)-(2cav'-a^2w-c^2u)\}]\frac{\alpha}{a}\right\}=0,$$

i.e. $$\Sigma[(cw'-bv')^2\{(2cav'-a^2w-c^2u)-(2abw'-b^2u-a^2v)\}$$
$$+(av'-cu)^2\{(2abw'-b^2u-a^2v)-(2bcu'-c^2v-b^2w)\}$$
$$+(bu-aw')^2\{(2bcu'-c^2v-b^2w)-(2cav'-a^2w-c^2u)\}]\frac{\alpha}{a}=0.$$

H. L. TRACHTENBERG.

198. [L¹. 3. a.] *A new method of finding the equation of the axes of the general conic.*

$$\lambda(ax^2+2hxy+by^2+2gx+2fy+c)$$
$$=[(ax_0+hy_0+g)x+(hx_0+by_0+f)y+(gx_0+fy_0+c)]^2$$

represents the system of conics having double contact with

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

at the points where it is cut by the polar of x_0y_0. Now if x_0y_0 is on an axis, and in that case alone, one of the conics of this system will be a circle, for the tangents from such a point to the conic are equal. Let λ_1 be the value of λ for this circle. Then the conditions for a circle give

$$\lambda_1a-(ax_0+hy_0+g)^2=\lambda_1b-(hx_0+by_0+f)^2,$$
$$\lambda_1h-(ax_0+hy_0+g)(hx_0+by_0+f)=0.$$

Thus, eliminating λ_1 the equation of the axes is

$$\frac{(ax+hy+g)^2-(hx+by+f)^2}{a-b}=\frac{(ax+hy+g)(hx+by+f)}{h}.$$

H. L. TRACHTENBERG.

199. [L¹. 3. a.] *Trilinear Coordinates.—The equation of the axes of the general conic.*

$$\lambda(u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta)$$
$$=[\alpha(u\alpha'+w'\beta'+v'\gamma')+\beta(w'\alpha'+v\beta'+u'\gamma')+\gamma(v'\alpha'+u'\beta'+w\gamma')]^2$$

is the system of conics touching the conic

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$$

at the points where it is met by the polar of $\alpha'\beta'\gamma'$. If $\alpha'\beta'\gamma'$ is on an axis, and in that case alone, one of the conics of this system will be a circle. Let λ_1 be the corresponding value of λ. Then writing down the conditions for a circle, a, b, c being the sides of triangle of reference

$$2bc[\lambda_1u'-(w'\alpha'+v\beta'+u'\gamma')(v'\alpha'+u'\beta'+w\gamma')]$$
$$-c^2[\lambda_1v-(w'\alpha'+v\beta'+u'\gamma')^2]-b^2[\lambda_1w-(v'\alpha'+u'\beta'+w\gamma')^2]$$
$$=2ca[\lambda_1v'-(v'\alpha'+u'\beta'+w\gamma')(u\alpha'+w'\beta'+v'\gamma')]$$
$$-a^2[\lambda_1w-(v'\alpha'+u'\beta'+w\gamma')^2]-c^2[\lambda_1u-(u\alpha'+w'\beta'+v'\gamma')^2]$$
$$=2ab[\lambda_1w'-(u\alpha'+w'\beta'+v'\gamma')(w'\alpha'+v\beta'+u'\gamma')]$$
$$-b^2[\lambda_1u-(u\alpha'+w'\beta'+v'\gamma')^2]-a^2[\lambda_1v-(w'\alpha'+v\beta'+u'\gamma')^2].$$

Putting each of these three equal quantities $=k$, and eliminating λ_1 and k from the three linear equations thus formed, we have for the equation of the axes

$$\left|\begin{matrix} 2bc(w'\alpha+v\beta+u'\gamma)(v'\alpha+u'\beta+w\gamma)-c^2(w'\alpha+v\beta+u'\gamma)^2 & & \\ \qquad -b^2(v'\alpha+u'\beta+w\gamma)^2, & 2bcu'-c^2v-b^2w, & 1 \\ 2ca(v'\alpha+u'\beta+w\gamma)(u\alpha+w'\beta+v'\gamma)-a^2(v'\alpha+u'\beta+w\gamma)^2 & & \\ \qquad -c^2(u\alpha+w'\beta+v'\gamma)^2, & 2cav'-a^2w-c^2u, & 1 \\ 2ab(u\alpha+w'\beta+v'\gamma)(w'\alpha+v\beta+u'\gamma)-b^2(u\alpha+w'\beta+v'\gamma) & & \\ \qquad -u^2(w'\alpha+v\beta+u'\gamma)^2, & 2abw'-b^2u-a^2v, & 1 \end{matrix}\right|=0.$$

H. L. TRACHTENBERG.

200. [L². 2. e; 4. a.] *A new method of finding the condition that the general conicoid should be one of revolution and of finding the equation of its axis.*

$$\begin{aligned}\{(ax_0+hy_0+gz_0+u)x+(hx_0+by_0+fz_0+v)y \\ +(gx_0+fy_0+cz_0+w)z+(ux_0+vy_0+wz_0+d)\}^2 \\ =\lambda[ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d]\end{aligned}$$

is the system of conicoids touching

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2fy+2wz+d=0$$

round the curve of section by the polar plane of $x_0y_0z_0$. If (1) the surface is one of revolution *and* (2) $x_0y_0z_0$ is on the axis of revolution, and under these conditions alone, one of the conicoids of this system will be a sphere, for all the tangents from $x_0y_0z_0$ to the conicoid are equal. Let λ_1 be the value of λ for this sphere. Then writing down the conditions for a sphere

$$\left.\begin{aligned}(ax_0+hy_0+gz_0+u)^2-\lambda a=(hx_0+by_0+fz_0+v)^2-\lambda b=(gx_0+fy_0+cz_0+w)^2-\lambda c, \\ (hx_0+by_0+fz_0+v)(gx_0+fy_0+cz_0+w)-\lambda f=0, \\ (gx_0+fy_0+cz_0+w)(ax_0+hy_0+gz_0+u)-\lambda g=0, \\ (ax_0+hy_0+gz_0+u)(hx_0+by_0+fz_0+v)-\lambda h=0.\end{aligned}\right\}$$

Thus, eliminating λ, the axis lies on all the following surfaces.

$$(a)\ldots\ldots\ldots\ldots \left\{\begin{aligned}&\frac{(hx+by+fz+v)^2-(gx+fy+cz+w)^2}{b-c} \\ &=\frac{(gx+fy+cz+w)^2-(ax+hy+gz+u)^2}{c-a} \\ &=\frac{(ax+hy+gz+u)^2-(hx+by+fz+v)^2}{a-b} \\ &=\frac{(hx+by+fz+v)(gx+fy+cz+w)}{f} \\ &=\frac{(gx+fy+cz+w)(ax+hy+gz+u)}{g} \\ &=\frac{(ax+hy+gz+u)(hx+by+fz+v)}{h}.\end{aligned}\right.$$

Now three of these surfaces are the three pairs of planes

$$(ax+hy+gz+u)\{h(gx+fy+cz+w)-g(hx+by+fz+v)\}=0.$$
$$(hx+by+fz+v)\{f(ax+hy+gz+u)-h(gx+fy+cz+w)\}=0.$$
$$(gx+fy+cz+w)\{g(hx+by+fz+v)-f(ax+hy+gz+u)\}=0.$$

Call these $\phi_1 . F_1 = 0$, $\phi_2 . F_2 = 0$, $\phi_1 . F_3 = 0$. If we can find one line and one only, that is common to all three pairs, that line must be the axis. Now the planes $F_1 = 0$, $F_2 = 0$, $F_3 = 0$ have a common line for $F_1 + F_2 + F_3 \equiv 0$ and *in general* no other three planes, one from each pair, have a common line. Thus in general the axis is given by

$$(\beta) \ldots\ldots\ldots f(ax+hy+gz+u) = g(hx+by+fz+v) = h(gx+fy+cz+w).$$

Putting each of these $= K$ and solving the equations we have

$$\frac{x}{\begin{vmatrix} h & g & u-K/f \\ b & f & v-K/g \\ f & c & w-K/h \end{vmatrix}} = \frac{y}{\begin{vmatrix} g & a & u-K/f \\ f & h & v-K/g \\ c & g & w-K/h \end{vmatrix}} = \frac{z}{\begin{vmatrix} a & h & u-K/f \\ h & b & v-K/g \\ g & f & w-K/h \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}}.$$

Thus the equation of the axis is

$$\frac{x - \dfrac{U}{D}}{A'/f + H'/g + G'/h} = \frac{y - \dfrac{V}{D}}{H'/f + B'/g + F'/h} = \frac{z - \dfrac{W}{D}}{G'/f + F'/g + C'/h}.$$ [*]

Also using the fact (β) that

$$ax+hy+gz+u : hx+by+fz+v : gx+fy+cz+w = 1/f : 1/g : 1/h$$

we get from (α)

$$\frac{1/g^2 - 1/h^2}{b-c} = \frac{1/h^2 - 1/f^2}{c-a} = \frac{1/f^2 - 1/g^2}{a-b} = \frac{1}{fgh},$$

$$\text{i.e. } b - c = \frac{hf}{g} - \frac{fg}{h} \text{ and two similars,}$$

$$\text{i.e. } \frac{gh}{f} - a = \frac{hf}{g} - b = \frac{fg}{h} - c$$

are the conditions for a surface of revolution. H. L. TRACHTENBERG.

Prof. E. B. Elliott's note (*Gazette*, Oct. 1905, p. 236) seems fallacious. To show this, suppose that the convergent sequence $s_1, s_2, \ldots$, of rationals tend to *no rational limit*. Then, given any positive rational ϵ', there is a (one or more) rational s such that, for some n and every p, $|s - s_{n+p}| < \epsilon'$. But, as ϵ' is diminished to zero, we can find *no* rational s valid for *all* such ϵ's. Thus the conclusion of (1) fails. The fact is, that logically a definition of a real number (*i.e.*, a sequence $s_1, s_2, \ldots$, of rationals such that, where ϵ is any positive rational, there is an integer n, which depends on ϵ, such that, for every p, $|s_n - s_{n+p}| < \epsilon$) precedes the criterion for the existence of a limit, and that then the sufficiency of this criterion follows at once (cf. *Encykl. der Math. Wiss.*, I. A 3, §§ 5, 6, 12, 13); though geometrical ways of thinking obscured this logical fact, even with Cauchy (cf. my remarks in *Bibl. Math.*, 1905, p. 206), until the time of Weierstrass.

PHILIP E. B. JOURDAIN.

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[*] The dashed capitals are minors of the corresponding small letters in $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

The undashed capitals are minors of the corresponding small letters in $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}$.

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ON HIGHER TRIGONOMETRY.

AN article by Mr. G. H. Hardy in the March number of *The Mathematical Gazette* (coupled with certain other articles in recent numbers) raises several points which can hardly be allowed to pass without further discussion. It is proposed to discuss these points under three different heads, of which the first two are treated here while the third will find space in a later number of the *Gazette*.

I.—THE "HIGHER" IN MATHEMATICS.

Mr. Hardy begins by asking the question: "How are we to define what we mean by 'Higher Trigonometry'?" and his answer is a somewhat superficial *description*, not the definition of a selective principle.

Might one be permitted to suggest that it is the notion of "a limit" that provides the dividing-line above which the 'Higher' Mathematics (which used only to be mentioned with bated breath by those outside the charmed circle!) appears; Calculus on the one hand, Higher Trigonometry on the other pointing the upward path to the cloudy regions of modern analysis? [Regarding these two fundamental branches of Higher Mathematics, the writer would like at once to register his opinion that the former is much better suited to a school-course, and that the latter "most fascinating and instructive region of mathematics" (Mr. Hardy, p. 284) provides meat essentially for strong men and not for babes. To take a personal illustration, the writer himself, at the age of 14 years, knowing little algebra, less trigonometry and absolutely nothing of infinite series, came under the tuition of a master (certainly a man of very striking character) who took up the Differential Calculus with a class of 20 boys (average age perhaps 16), and by means of it gave him an interest in all parts of the subject which culminated in a permanent connection with mathematics. The experiment was altogether a great success, and some such course, under a different master, sends to our university from that school the youths best equipped for mathematical work.]

Starting then with the above-mentioned principle of separation, some account of the logical evolution of the limit-conception does not seem out of place, especially since such an account is not easily to be found; even in the 2nd edition of Chrystal's Part II. the matter is scattered and to some extent incomplete.

The term *limiting value* is originally introduced on account of the fact that "a function of x" may have a value for a given value (a) of the argument, although "the expression," $f(x)$, fails to provide a value when a is substituted for x in it; this is because the function is regarded as having an identity of its own, apart from any particular expression of it (a fact which is associated with the possibility of expressing it in a variety of ways).

An expression $f(x)$ is the symbolical embodiment of instructions by the aid of which the value of the function corresponding to a given value of the argument, may be obtained; but cases arise in which the processes indicated cannot be carried out, *e.g.* the expressions $1/(x-a)$, $\sin(x-a)/(x-a)$ do not provide a value when x has the value a. In such a case it may be that another expression, algebraically equivalent to $f(x)$ when both give values, gives a value when x has the value a, *e.g.* the value $+1$ of the function expressed by $\sin(x-a)/(x-a)$ may be obtained from the expression $\{1-\theta(x).(x-a)^2/3!\}$ in which $\theta(x)$ has positive proper fractional values for all real values of x. In all such cases it may be said that the expression $f(x)$ establishes a *continuum of values* which belong to the function, and that there is one number (l) which, if it be regarded as corresponding to the value a of the argument, will *complete* this continuum: $f(x)$ is then said to have l as a limiting value for the value a of x, or to "tend to l as x tends to a"; and the analytical test of continuity for the case of a real function, demands that: corresponding to any assigned finite positive real number (ϵ), however small, there must be a finite positive real number (h) such that $|f(a+\theta h)-l|<\epsilon$, θ being a variable whose values lie between -1 and 0, between 0 and $+1$, or between -1 and $+1$. This test does not provide information as to the *existence* of a limiting value; and the comparative simplicity of the Differential Calculus depends on the fact that in the limit-problems raised by it, the limits where they occur can be actually evaluated, *i.e.* the one number which sustains the test can be obtained by elementary algebraic methods.

The notion of *limiting value* is, then, primarily associated with the values of a *dependent* variable; but the logical extension of the conception demanded by one particular type of function provides the useful symbols $\pm\infty$ which *can only be regarded as limiting values*, and yet may be given as values to an *independent* variable. Simple examples will indicate more clearly what has necessarily to be stated somewhat vaguely in its general aspect:

Consider (i) the function determined by the two equivalent expressions $(x^2-1)/(x-1)$ and $(x+1)$. With reference to the former expression the value $+2$ (obtained from the latter) corresponding to the value $+1$ of x, is said to be a limiting value; the peculiarity is, however, in no way associated with the abstracted symbols $+1$, $+2$, but resides wholly in the mode of expressing the function under consideration. If, on the other hand, (ii) $f(x)=1/(1-x)$ we write $f(1-0)=+\infty$, $f(1+0)=-\infty$, meaning that the continuum of functional values determined by values of x less than $+1$ is such that no positive real number can be chosen so great that it does not belong to this continuum, etc.; it is then said that "$f(x)$ tends to infinity as x tends to $+1$." *But no matter which of its expressions is used, the function cannot be regarded as having, when $x=a$, a* **value** *in the original sense,*

i.e. $\pm\infty$ are *essentially limiting values.*

We are now logically compelled to admit $\pm\infty$ to positions among the symbols belonging to the real number continuum and to consider the possibilities that arise from permitting $\pm\infty$ to be given as (limiting) values to a function-argument. [In the Dedekind-scheme $\pm\infty$ will have *schnitts* in which all the rationals are relegated to one class, leaving none for the other.]

The formula $f(\infty)=l$, like $f(a)=\infty$, represents essentially a limit-statement—the burden being in this case thrown on the *independent* variable; and this statement is expressed in finite terms as follows: corresponding to any assigned finite positive real number (ϵ), however small, a finite, positive real number (p) can be found such that $|f(x)-l|<\epsilon$ for all values of x greater than p.

There is, however, this other peculiarity about the tending of the argument to either of the limiting values possible for it, that the corresponding functional variation cannot be subjected to anything like the same closeness of scrutiny as in any finite range of argument-values; and although the case in which x tends to infinity by continuous variation can be (and generally is), reduced to the ordinary case by a "substitution," *e.g.* $x'=1/(x-a)$, a further extension of the limit-conception clearly demands consideration, to cover the possibility of a function having what may without violence be called "a limiting value" when its argument tends to infinity by a system of values which do not form a continuum: in particular to meet the important case in which the expression $f(n)$ has only a meaning for *integral* values of n, *e.g.* if $f(n)=1-\frac{1}{2}+\frac{1}{3}\ldots+1/(2n-1)$.

The notion of limiting value is thus in this one ultimate case separated from its dependence on continuity*; and this separation marks broadly the dividing-line between the two fundamental branches (Differential Calculus and Higher Trigonometry) of Higher Mathematics.

The distinguishing feature of Higher Trigonometry is its concern with "infinite sequences," *i.e.* with the variation of functions when the argument tends to infinity by *integral* variation; and the essential difficulty of this branch of mathematics is to be found in the fact that, in general, the infinity of the argument cannot with advantage be got rid of by a substitution such as $x'=1/(x-a)$, and that, in consequence, algebraic transformations of the functional expression do not provide a simple means of determining the limiting value when it exists (and so of establishing its existence). Such cases as that of an infinite series of terms in geometric progression are quite exceptional, on account of the fact that the finite series can be "summed"; the treatment of the infinite series thus having the same kind of simplicity as the demonstration of $D.x^m=mx^{m-1}$, because a "closed" expression can be found which is equivalent to the "unclosed" expression $f(n)$; it is, therefore, suitable for inclusion in a school-course. But this does not, in the writer's opinion, provide a satisfactory starting-point for the theory of infinite series; and the summing to infinity of the geometric series can only be regarded as having an interest in connection with Recurring Decimals, until its importance to that theory is made clear.

Some method then has to be taken for the purpose of introducing the student gently to the theory of Infinite Series; and it would appear that this is most suitably to be found through the Circular and Exponential functions. If this be granted, the method (which has found much favour since it was invented for the Binomial Theorem by Euler) advocated by Mr. Hardy $\left[\text{cf. (B), that } \left(1+\frac{1}{1!}+\frac{1}{2!}+\ldots\right)^x=\left(1+x+\frac{x^2}{2!}+\ldots\right)\right]$ must be considered "illogical"—besides that it is perhaps open to a charge of "artificiality"; these, however, are hard words, which may not be used with confidence until the mathematical circle has been rounded off. Just to take one point, the use of this method leads Mr. Hardy to the early declaration that "*irrational* powers should at this stage be severely and *explicitly* left alone"—yet the fact that it satisfies $D.y=y$ is its simplest fundamental property! Indeed this equation provides perhaps the most "natural" way of introducing the function; and, apart from the variety of other criticisms possible, *it* does not throw into relief the exponential characteristic.

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* The reason for this somewhat astonishing dissociation of two fundamentally interrelated conceptions is perhaps to be found in the fact that, in the very nature of things, mathematics can only deal with continuity by discrete approximation.

It remains to develop in a rigorous, and at the same time satisfactory, way the old process—which Mr. Hardy repudiates; taking the function $(1+z/n)^n$ as in itself an interesting case for binomial expansion, dealing only with integral values of n so as to avoid worse things, and obtaining thus the infinite series in a straightforward way which exhibits its exponential relations. The objections to such a process really hinge round the difficulty that is found in establishing, at this stage of the student's course, theorems of existence for limiting values, and this difficulty has now to be discussed.

II.—LIMIT OF A CONVERGENT SEQUENCE.

IF the function $f(n)$ tends to a limit as n tends to infinity by positive integral variation, it is quite simply shown that corresponding to any assigned ϵ, a value (N_ϵ) of n can be obtained such that $|f(N_\epsilon+p)-f(N_\epsilon)|<\epsilon$ for all positive integral values of p.

The converse, viz., that if corresponding to any assigned finite positive real number (ϵ), however small, a finite value (N_ϵ) of n can be determined such that $|f(N_\epsilon+p)-fN_\epsilon|<\epsilon$, $f(n)$ tends to a limit, *i.e. the criterion for the existence of a limit to the sequence* is by no means so obvious; and one need have little hesitation in asserting that its abstruseness of demonstration has been the primary cause of the obscurity in which the fundamentals of Higher Trigonometry have remained: witness recent articles in the *Gazette.*

It is interesting to nŏte that in the first edition of Chrystal's *Algebra* [chap. xxvi., § 3] the proof given for the sufficiency of this criterion is only a proof of the fact that *there cannot be two different limiting values*, and that in a work of so thorough a character the second edition merely provides an apologetic footnote confessing to want of rigour—not making the necessary changes, but giving a reference to the section on number-theory which has been included in that edition! This is in itself sufficient evidence of incongruity in the means employed to obtain rigour in so fundamental a thorem. Hobson follows Chrystal in this particular laxity [*Trigonometry*, § 193, chap. xiv.]. Whittaker, who draws particular attention to the importance of the theorem [*Analysis*, § 6] has a proof which is heavily encumbered with algebraic detail and which only in the end succeeds in reducing the general theorem to what the writer of these remarks regards as merely a more intuitively obvious case [see Theorem III. below]. Dedekind himself, without whose number-theory no simple treatment of higher mathematics seems possible, made a courageous attempt to apply his theory directly to the problem under consideration,[*] and his demonstration has been reproduced in various places;[†] but the argument is not convincing, and the present writer at least is forced to the conclusion that the fact has been implicitly used in the proof of it. It is thus particularly gratifying that this most persistent obstacle has been removed in the pages of the *Gazette* [Oct. 1905, p. 236] by Professor E. B. Elliott, who undoubtedly went right to the heart of the matter when he applied the method of proof that has been, at least since the days of Euclid, associated with converse-theorems; his double application of the *Reductio ad Absurdum* cannot fail to strike everyone with admiration. One of his readers, however, disputes the validity of his preliminary theorem (1) [*Gazette*, May 1906, p. 327]; but Mr. Jourdain himself provides the explanation of his own error in the implication that a certain order of ideas is obligatory, when he states that "logically a definition of a real number (*i.e. a sequence* $s_1, s_2, \ldots$ *of rationals such that, where* ϵ *is any positive rational, there is an integer* n, *which depends on* ϵ, *such that for every* p $|s_n - s_{n+p}| < \epsilon$) precedes the criterion for the existence of a limit;" the italics have been supplied to indicate the abstruseness of the number-theory which has prevented the possibility of a sound elementary treatment of higher mathematics. All that need be said more is that Dedekind has waited long for the appreciation of his discovery of the arithmetical "essence of continuity," which made Nov. 24, 1858, the red-letter day of his life [cf. Essay I., Introduction]! The specification of a real number requires only a *schnitt* of the rationals, not a convergent sequence; the continuum of real numbers comes first, according to the views expressed in Section I. of this paper, and convergency is a conception derived from that of continuity.

The following presentation of the limit-theorems is based on Professor Elliott's theorem (1), the enunciation and demonstration of which given here have his approval:

I. *If the function* f(n) *does not tend to a limit as* n *tends to infinity, a finite positive real number* (κ) *can be determined such that whatever value* z *has, and whatever positive integral value* n *has, there is at least one positive integral value of* P_n *for which* $|f(n+P_n)-z| > \kappa$.

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[*] *Essays on Number*, by Richard Dedekind (Open Court Publishing Co., Chicago), I. vii., p. 24.

[†] Recently by Fine, *College Algebra* (Ginn & Co.), § 197, p. 61.

Proof. If this is not so, then there must be at least one value (l) of z, and a positive integer (N), such that for *all* positive integral values of p $|f(N+p)-l|<\epsilon$, *however small a finite positive real value be given to* ϵ;
i.e. $f(n)$ tends to the limiting value l, which contradicts the hypothesis.

II. *If* f(n) *is a real function which increases* (*decreases*) *continually as* n *tends to infinity, but remains always less* (*greater*) *than a certain specified real number* (a), f(n) *tends to a limiting value which* $\not>$ ($\not<$) a.

Proof. Confining our attention to the case of an increasing function, let us suppose that there is no limiting value; then by Theorem I., a finite positive real number (κ) can be found such that for this simple case $f(n)<a-\kappa$ for all values of n; hence, by a second application of the theorem, $f(n)<a-2\kappa$ for all values of n, and so on; *i.e* $f(n)$ must be a continually increasing function which remains always less than any assigned real number—which is absurd.

III. *If* f(n), F(n) *are two real functions such that* f(n) *increases continually and* F(n) *decreases as* n *tends to infinity, if* F(n) *has no value which is less than any value of* f(n), *and if* F(n)$-$f(n) *can be made as small as we please by choosing* n *sufficiently great, the two sequences determine a single real number which is the limiting value of each function.*

Proof. If $a=F(a)$ where a denotes a finite value of n, the function $f(n)$ satisfies the requirements of Theorem II.; hence $f(n)$ tends to a limit (l), and similarly $F(n)$ tends to a limit (L). Suppose, if possible, that ($L-l$) is finite, say δ (clearly positive); by hypothesis we can choose a value (N) of n such that $F(N)-f(N)<\delta$: but $F(N)>L$ and $f(N)<l$—which is absurd.

Finally, and most generally,

IV. *If corresponding to any assigned finite positive real number* ϵ, *however small, a finite positive integer* (N_ϵ) *can be found, such that* $|\mathrm{f(m)}-\mathrm{f(n)}|<\epsilon$ *provided only that* m, n *have integral values greater than* N_ϵ; f(n) *tends to a limit* [cf. Professor Elliott's Theorem (2)].

Proof. If there is no limit, then (by Theorem I.) a finite positive real number (κ) can be found such that

$$|f(n+P_n)-f(n)|>\kappa,$$

there being at least one positive integral value of P_n for each value of n. But, by hypothesis, if $n>N_\kappa$, $|f(n+p)-f(n)|<\kappa$ for *all* positive integral values of p. These inequalities being contradictory, it follows that $f(n)$ must tend to a limiting value.

These theorems form a sound elementary basis for the work of the higher analysis, being fundamental to the theory of Infinite Series and to the Integral Calculus. D. K. PICKEN.

MATHEMATICS FOR ARMY CANDIDATES.

During the last few years the methods of teaching Mathematics in schools have been subjected to much adverse criticism: this has induced teachers to apply themselves to the task of setting their house in order. It is now generally deemed advisable to give greater prominence to practical applications, to make more use of accurate drawing as a method of solution of problems and for purposes of verification, to curtail numerical calculations by dispensing with figures of doubtful or unnecessary accuracy, and to lay more stress on the employment of graphical methods.

These improvements have been advocated by the Committee of the Mathematical Association, and few will be disposed to deny that, apart from any advantages of the much vaunted useful order, such changes render the subject more attractive to the average boy than was the case when the connection between theory and practice was not always clearly shown.

If these reforms were carried out, as they were so intended, by the teachers themselves, and due attention were given to the points which in the past may have been unduly neglected, the educational efficacy of Mathematics would no doubt be increased, and the pupil would not only obtain a better grasp of the theory, but would be better able to apply it in practice. But behind the schoolmaster is the examiner; and, unfortunately, the views of some educational authorities by no means coincide with the programme of reforms propounded by those who wished to see the teaching carried out on more practical lines.

The policy of "overturn, overturn" has been pushed too far: if numerical calculations, graphical representations, and geometrical drawing are not merely to be an aid, but a substitute for a systematic knowledge of the theory of Elementary Mathematics, the value of the subject as an educational instrument will be but small.

To show that there is a real danger of such a revolution, we wish to draw attention to the class of knowledge which is now demanded of candidates for commissions in the army.

There may be different opinions as to the educational value of a subject taught according to any given syllabus, but every one will agree that, when examining bodies demand knowledge of a special kind, their requirements should be stated as carefully and precisely as possible. That the knowledge required by the Civil Service Commissioners is of a very special kind will be readily conceded, but the language in which their wants are formulated leaves much to be desired, though the statements of what they do not require sometimes form oases of the blest and the intelligible.

Under the heading of Arithmetic, for the Qualifying Examination, we find "mensuration of plane figures and solids." Surely a departure of this magnitude should be defined a little more carefully. The Algebra is "to simple quadratic equations"; but "skill in elaborate analysis will not be looked for." In the list of subjects for the Competitive Examination the Trigonometry required is "up to and including solution of plane triangles"; but nothing is said as to formulae involving two angles or identities, and we are left to derive what comfort we may from the usual footnote disclaiming any demand for "great analytical skill." The schedule of Geometry issued by the University of Cambridge is used to define the Geometry; this, so far as the syllabus is concerned, is satisfactory, and we wish that similar means had been employed in other cases. The Algebra for Math. II. includes "a working knowledge (without rigorous fundamental demonstration) of the elementary infinite series for $(1+x)^m$, e^x, $\log(1+x)$, $\sin x$, $\cos x$, $\tan x$."

The Differential and Integral Calculus is to include "simple applications to the properties of curves, to turning values, and to easy mechanical and physical problems." This is the first time we meet with the word "easy"; one could wish for some clearer indication as to its meaning in this case. The Coordinate Geometry opens magnificently by demanding "straightforward applications to the straight line, circle, ellipse, parabola, hyperbola, cycloid, catenary, logarithmic spiral, and other curves of common occurrence." But we are informed later that "a systematic knowledge of conics is not required." The programme concludes with "the elementary statics of liquids and gases."

We have quoted to this extent from the syllabus to justify our complaint that it is lacking in that definiteness, which teachers have a right to expect from their dictators, but more especially to show that the qualifying negatives, with which the instructions bristle, and the extensive range of subjects demanded from boys, who cannot specialise in mathematics, indicate that the framers of this syllabus expect the subject to be dealt with on a new and original plan. We are more anxious to find out what is actually required than desirous to criticise the language employed; we therefore turn to the sole supplementary source at present available, the papers set in November, 1905.

The questions set involve little else but numerical calculations and graphic work. There is, however, one notable exception, which is the following. "*P*, *Q*, *R* are the mid points of the sides of a triangle. Construct the triangle and prove the theorem on which your construction depends." The last part of this question seems to be a sort of a riddle without an answer, but we are glad to note such an explicit admission of the existence of theorems.

In the two papers set for Math. I. the subjects are represented by the following numbers of questions: Arithmetic, 2; Geom. Drawing, 3½; Theoretical Geometry, 1½; Algebra, 0; Graphs, 2; Theoretical Trigonometry, 0; Numerical solution of Triangles, 3; Statics (Graphical), 2½; Statics (Theoretical), 1½; Dynamics, 0; Mensuration, 2. From this analysis of the papers it is abundantly evident that the examiners do not require a sound theoretical knowledge of any subject, but only the ability to apply special formulae and methods to the numerical solution of a certain class of questions which they regard as of practical importance. Questions are also set involving measurements and weighing; these are dignified by the title of "Practical Mathematics." On this point it is useless to enlarge; the great merit of mathematics as an educational subject is to teach deductive reasoning; whether the educative effect of these experiments is proportional to the time spent on them is open to question, but as a part of mathematics they seem somewhat out of place.

The papers set for Math. II. further confirm the impression hitherto made, and we are forced to the reluctant conclusion that the candidates are expected to be in possession of a few knoblets of knowledge which will enable them to solve questions of a strictly practical and "useful" kind.

If this is really the aim of the army authorities, if they desire mechanical manipulation of formulae rather than intelligence; in short if they prefer the less to the greater they will assuredly reap what they sow. F. E. Robinson.

ON A SUPPOSED SOLUTION OF THE "FOUR-COLOUR PROBLEM."

It appears from the recently published life of Archbishop Temple that he is supposed to have found a proof of the proposition that any map can be coloured with four colours, no two adjacent segments of it being coloured alike. The writer had heard of this before the publication of the Life, but had not succeeded in finding what the proof was, and was under the impression that it had never been published, especially as it was not mentioned in Mr. Rouse Ball's book. As stated in the Life the supposed solution was published in the *Journal of Education*, June 1, 1889. It turns out to be what one has learned from experience (and the writer owns that he has spent much time on the subject) to expect from any theory which looks like a short solution of this extremely elusive problem.

In beginning the investigation one comes at once upon simple principles which promise to contain the key, *e.g.* that the belt of segments surrounding a given segment only requires at most three colours beside that of the given segment: but such principles are only immediately applicable to a figure constructed on some given uniform plan, and in the attempt to extend them to every mode of construction all kinds of complications and difficulties arise.

Temple's theorem is just of this character. His investigation stops before the real difficulties begin, and it may be suspected that many of those who have tried the puzzle have got as far as he has, either by arriving at the self-same theorem, or something analogous to it, and similarly related to the whole problem.

In fact Temple's principle occurred to the writer in a simple self-evident form at an early stage in his own attempts, but he soon satisfied himself that it was no general solution.

The theorem Temple proves is this:—

"There can be four figures on the same plane, no one of which shall have a common part with any other, and every one shall be partially conterminous with every other, but there cannot be more than four."

The essential part of this had occurred to the writer in the following form:—

"If three segments of a map are adjacent each to each, then if a fourth segment is adjacent to each of them, one at least of the first three must be completely enclosed."

The theorem in either form is only a complete solution for a figure constructed as follows: make three segments adjacent each to each; add another adjacent to each of the first three, and leaving therefore at most two unenclosed. Treat the external three segments which may thus result in the same way by adding another adjacent to them. If only one of the original three is unenclosed, there will be two external segments; add another adjacent to both, and treat these three like the first three. Similarly if all of the first three are enclosed.

The theorem could only apply immediately to every figure under certain presuppositions which are certainly not true of every figure.

(i) It would have to be presupposed that if three segments necessarily have three different colours they must be adjacent each to each. But this raises the question whether three segments not thus adjacent might necessarily have three different colours because of their relation to other parts of the figure. And it is easy to shew that this may be the case. It would be necessary therefore to prove that three such segments can only arise in such a way that either (1) not more than one other segment can be adjacent to all three, or (2) that if there could be two others adjacent to all three, these last two cannot be adjacent to one another. This would be a new and difficult investigation and require new considerations.

(ii) It would have to be presupposed also that if a segment D_1 is adjacent to three segments A_1, B_1, C_1, which must have three different colours A, B and C, and so must itself have a fourth colour D, any other segment X_1 adjacent to D_1 could only be determined to a colour different to A, B and C by being adjacent to the same set A_1, B_1 and C_1. But it would have to be considered whether there might not be another set of three segments which necessarily have the same three colours A, B and C; and it can be shewn again that this is possible.

If then we call such a set A_2, B_2, C_2, we should have to shew that it could only arise in such a way that either (1) no other segment could be adjacent to every member of it, or no other than D_1; or (2) that if a segment X_1 could be adjacent to A_2, B_2 and C_2 it could not be adjacent to D_1. (Other combinations also are possible.) These again are new and difficult investigations requiring, as before, new considerations.

Further, if a map is sufficiently complicated, it will often be found on trial, that if we colour it continuously starting from a segment taken at random, and never introduce a new colour till it is necessary, we shall nevertheless, in closing upon a part of the figure already coloured, get segments of the same colour adjacent to each other, and an alteration of some of the colouring done will be required.

A really satisfactory solution therefore ought to give a definite rule of procedure by which any map whatever can be coloured as required with four colours only. The supposed solution before us clearly does not give any such rule. J. COOK WILSON.

ILLEGITIMATE DIFFERENTIATION.

IT has been my experience, and probably that of most teachers, that even advanced students who ought to know better cannot differentiate the simplest algebraic and trigonometric functions without indulging in an extravagant luxuriance of infinite series and making a number of illegitimate assumptions.

In a note in the *Mathematical Gazette* for May, Prof. Bromwich calls attention to the objectionable use of infinite series in differentiating powers. I find that students, who have gone through an extended course of higher mathematics, cannot correctly differentiate even such simple expressions as $\sqrt{x}$ or $\sin x$. In the latter case they put $\sin dx = dx$ and $\cos dx = 1$, regardless of the fact that the latter assumption gives rise to a term which ultimately takes the form $(1-1)\div 0$, to say nothing more.

It is important that beginners, who are going on to courses in engineering or physics, should learn the meaning of a differential coefficient and an integral long before they are introduced to the theory of convergence and divergence, and should not be allowed to use infinite series until they can be carefully told when the use of such series is legitimate. There is no reason why the study of the Calculus should be postponed till after the Binomial Theorem has been learnt for a positive integer, not to mention for a fraction.

I have examined several text-books with the intention of recommending them to engineering students beginning the Calculus, but have found none which supply exactly what is required. Most text-books are so full of $e^{\sin^{-1}x}$ and log tan $\frac{1}{2}x$ and similar expressions that the principles of the Calculus itself are quite hidden from notice.

The proper way of beginning the Calculus is by applying it to simple rational algebraic expressions such as $2x^2-3x+4$. The rules for the differential coefficient of a sum and product should then be given, and the latter illustrated by verifying it in the case of a product of two linear factors such as $(3x-1)(2x+5)$.

The differential coefficient of a power then follows at once by repeated applications of the product rule. When the beginner has got so far he can learn to find the slope of the graph of any simple rational algebraic expression, and to work other illustrative examples such as calculating velocity at any instant from the formula for the space described, and by this time he will be familiar with the nature and meaning of a differential coefficient.

(*a*) If desired he can then learn the rule for the differential coefficient of a quotient, which leads to the differentiation of negative integral powers and simple fractions.

When it is necessary to differentiate powers with fractional indices (and up to this stage no knowledge of the theory of fractional indices is needed), the student should be introduced to the rules for the differential coefficient of a function of a function

$$\frac{dy}{dx}=\frac{dy}{dz}\frac{dz}{dx}=\frac{dy}{dz}\Big/\frac{dx}{dz},$$

and the differential coefficient of $y=x^{p/q}$ obtained by putting $x=z^q$ and $y=z^p$ in this rule.

(*b*) There is no reason to begin (*a*) before introducing the Integral Calculus. The areas of the graphs of simple rational algebraic expressions afford abundant exercises in integration, and the calculation of areas and possibly volumes is interesting to beginners. Theoretically speaking there is no reason why a beginner should not even learn to calculate moments of inertia in simple cases before he has learnt to differentiate a quotient. Whether this plan would be expedient it is not our object to discuss. At any rate the volumes of pyramids might be found.

Most beginners have very hazy ideas regarding the why or wherefore of the quantity e as a base of logarithms. Now e need not be introduced until its use is made obvious by trying to differentiate with respect to an index, and it is then conveniently introduced in the following manner:—If we try to differentiate $y=a^x$ we obtain by the rule,

$$\frac{dy}{dx}=a^x \times \operatorname*{Lt}_{h=0} \frac{a^h-1}{h}.$$

The slope of the curve $y=a^x$ is thus proportional at any point to the ordinate y, and a rough idea of the above limit can be got by drawing the graph. The limit represents the slope of the graph at $x=0$.

Now let e be a quantity defined by the property that the slope of the graph of the curve $y=e^x$ when $x=0$ is equal to unity; we then have

$$\frac{de^x}{dx}=e^x.$$

To show that the definition leads to a finite value of e, draw the graphs of $y=2^x$ and $y=4^x$. In the former, the slope of the chord joining $x=0$ and $x=1$ is unity, the slope of the chord joining $x=-1$ and $x=0$ is $\frac{1}{2}$, hence the slope of the tangent at $x=0$ is between $\frac{1}{2}$ and 1. In the latter curve, by drawing the chord joining $x=-\frac{1}{2}$ and $x=0$, and the chord joining $x=0$ and $x=\frac{1}{2}$, the slope of the tangent at $x=0$ is shown to lie between 1 and 2. There must, therefore, be a value of a between 2 and 4 such that the slope of the tangent of $y=a^x$ at $x=0$ is equal to unity.

The differentiation of a^x now follows by putting $a=e^n$, or $n=\log_e a$, giving

$$\frac{da^x}{dx}=\frac{de^{nx}}{dx}=\frac{de^{nx}}{dnx}\frac{dnx}{dx}=ne^{nx}=a^x\log_e a,$$

and from this starting point the whole of the bookwork associated with the differentiation and integration of exponentials and logarithms can be done without once introducing an infinite series.

There are other ways of approaching the subject, and of these, perhaps, some method of defining natural logarithms without reference to a base in the first instance, using the notation lognat x, may be the best. The definition of logarithms by means of a base in the elementary teaching of mathematics is open to many objections. It tends to make the beginner think that when mathematicians had practically an unlimited choice of convenient simple *commensurable* numbers which they might take as their base, they chose an inconvenient incommensurable number as base, as the schoolboy said "to make it harder for us." The *value* of the discovery of logarithms had little to do with the base, and depended entirely on the property that the numbers discovered enabled arithmetical calculations to be simplified by replacing multiplication by addition. If then logarithms are defined by the multiplication theorem it is seen that when one such set of numbers have been found, every set of equimultiples of these numbers possesses the same property; in other words that different systems of logarithms only differ by a constant factor. How many beginners fully realise this? In the common logarithms the constant factor is chosen to make $\log 10 = 1$, and the *base* of any system is merely the number whose logarithm is unity.

If we define lognat x as being equal to $\int dx/x$ between 1 and x, in other words to the area of the hyperbola $y = 1/x$ measured from $x = 1$ it is easily shown that lognat x is a function satisfying the multiplication theorem; probably a more convenient way would be to define lognat a as the limit of $(a^h - 1)/h$ so that

$$\text{lognat } a = \frac{1}{a^x}\frac{d}{dx}(a^x),$$

then the rule for the differentiation of a product gives at once lognat $(ab) =$ lognat $a +$ lognat b, and the rule for differentiation of a power gives lognat $a^n = n$ lognat a showing that natural logarithms possess the same properties as ordinary logarithms The natural logarithm of any number a could then be constructed graphically by measuring the slope of the graph of a^x, and the base e of the natural logarithms thus defined.]

The mistakes that commonly occur in differentiating sines and cosines arise from $\sin(x+h)$ being expanded by the addition formula instead of $\sin(x+h) - \sin x$ being expressed as a product. This mistake can be avoided in an instructive way by obtaining the differential coefficient from the limit of

$$\frac{f(x+h) - f(x-h)}{2h} \text{ instead of } \frac{f(x+h) - f(x)}{h}.$$

I wonder how many people there are who know that the former fraction gives a better approximation to $f'(x)$ than the latter which is commonly used in defining $f'(x)$. It is not necessary to know Taylor's series to see that such is the case, it may be seen geometrically by drawing the corresponding chords and tangents to any simple graph.

The second mistake is made when $\sin h$ is put equal to h or $\sin \frac{1}{2}h$ equal to $\frac{1}{2}h$ instead of $\sin h/h$ being put equal to unity.

To a person with extended mathematical knowledge either mode of expression is equally significant, but to a beginner the statement $\sin h = h$ is hardly likely to convey more meaning than that because $\sin h$ and h are both zero, one is equal to the other, which is incorrect. To understand the distinction the study of the Calculus forms the best preparation, therefore the statement should not be made at the earliest stage of that study.

It is for this and other reasons desirable that the differentiation of sines and cosines should be proved in the first instance independently of circular measure in the form

$$\frac{d}{dx}\sin x = \cos x \times \underset{h=0}{\mathrm{Lt}} \frac{\sin h}{h},$$

so that the differential coefficient of $\sin x$ is recognized to be *proportional* to $\cos x$ whatever be the unit of angular measurement. That the limit is finite is shown by the fact that the graph of $\sin x$ has a finite slope.

The necessity for this treatment is shown by the fact that science student seven in the second or third year of a university course often *cannot give even the correct answer* in differentiating the sine of an angle expressed in degrees.

What has been said about e applies with equal force to the premature introduction of circular measure into elementary courses. Why mathematicians should be so silly as to revel in a unit of angular measurement which makes all angles that are of any use incommensurable, must be a puzzle to anyone who thinks for himself, and must deter many boys from applying common sense to mathematics. Practically the use of circular measure is not obvious till we try to differentiate sines and cosines.

By confining attention in the first instance to rational algebraic functions, Taylor's and Maclaurin's Theorems may be introduced without the use of infinite series, and may be applied to such exercises as finding the result of substituting say $x = 2 + y$ in $x^3 - 3x^2 + 5x + 2$. These might very well be treated simultaneously with the Binomial Theorem for a positive integral index.

It is surprising that the late Mr. R. A. Proctor in his interesting little book did not differentiate a quotient without introducing an infinite series of "higher powers which may be neglected."

In conclusion, I would not introduce a single infinite series into Elementary Calculus until perhaps the very end when it might be safe to put a few of the expansions of the principal functions (including the Binomial Theorem) into the hands of the learner. Infinite series are dangerous tools which in many cases complicate rather than simplify. They ought to be tabooed from the elementary teaching of mathematics, together with the

base e, the radian, and the poundal, not to mention the still worse "engineers' unit of mass"; and any candidate making use of these means of computation, without reasonable and sufficient cause shown, should be liable for each offence to a penalty not exceeding FIVE MARKS. G. H. BRYAN.

THE DISCUSSION OF CERTAIN POWER-SERIES.

THE difficulties which arise in teaching the elementary power series appear to form part of a larger question, namely, the proper course of mathematical education after a sufficient knowledge of Elementary Geometry, Algebra and Trigonometry has been acquired. The question of Geometry is hardly germane to the present discussion, and I will only mention it so far as to say that I believe it is the hope of many teachers of mathematics that boys of 15 or 16 may in the future learn far more of Projective and Descriptive Geometry than is the case at present.

I imagine that the average student of thirty years ago went through the whole of his University course without any doubt as to the validity of proofs which are now generally regarded as inaccurate. At a somewhat later date such a student imbibed these proofs in the earlier part of his education only to find a year or two later that he had been deceived, and he thus acquired an unsettled feeling as to the validity of much of his previous knowledge. At the present time he is, in most cases, faced with proofs of great complexity, without any idea of the underlying principles, and probably ends by more or less consciously cramming them for examination purposes. The question which I wish to raise concerns the nature of these principles and the best method of expounding them.

The modern analysis appears to be based primarily on two concepts, number and function, and I believe that those who are better qualified than myself to discuss this matter would say that no accurate understanding even of the elementary series can be attained without previous knowledge of these concepts. On the other hand, if the scholarship candidates next autumn were compelled to write short essays on "Incommensurable numbers and the methods of operating with them," and "The meaning of the term function with illustrative examples of different types," no one will doubt that the average attempt would be unsatisfactory. I have therefore, in my own teaching, attempted to deal with these questions first, and will now proceed to give the following summary of the course, not as claiming that it is or approximates to a perfect solution, but rather with the hope that it may serve as a basis for discussion of the nature recently invited by the Editor.

Incommensurable numbers are treated in the method of Jordan in his "Cours d'Analyse" but entirely on a numerical basis, and the theory of limits in general follows naturally. Concrete examples include surds, the calculation of logarithms by successive approximation, the length and area of a curve. Meantime a series of curves are drawn by the students from numerical calculation, these being selected so as to show that the singularities discussed later are real and such as may actually occur in functions already considered. They include

$$y=(1+x)^{\frac{1}{x}},\ y=\frac{10^x-1}{x},\ y=x^n(10^{\frac{1}{x}}-1)/(10^{\frac{1}{x}}+1) \text{ for } n=0,1,-1,$$

and $$y=x^n \sin\frac{\pi}{x} \text{ for } n=0,1,-1.$$

The general definitions of a function and its continuity are then discussed at some length, of course with graphical illustrations. Curves are drawn at random on squared paper and the values of the range ($\pm h$) of x for a given maximum variation (ϵ) are plotted, thus introducing the idea of open and closed ranges for x and the uniform continuity of the function in a given range.

The differential coefficient is then considered—with illustrations depending largely on the individual students—and the differential coefficients of simple algebraic and trigonometrical functions are found, that of x^n being deduced from $\underset{t=0}{dt}\,\frac{t^n-1}{t-1}$. Then follow the theorem of mean value and simple properties of the differential coefficient, and *from these* are deduced separately, as easy examples, the inequalities

$$\frac{x^p-1}{p} \gtrless \frac{x^q-1}{q} \quad \text{and} \quad ma^{m-1}(a-b) \gtrless a^m-b^m \gtrless mb^{m-1}(a-b).$$

The differentiation of a^x is then attempted (a being of course positive), and is found to depend on $\underset{h=0}{\mathrm{Lt}}\,\frac{a^h-1}{h}$. This is proved, by the above inequalities, to be a number lying between $\frac{a-1}{a}$ and a, *independent of the manner in which h approaches zero* and therefore only dependent on a. It is therefore a function of a, say $l(a)$, defined uniquely for all positive values of a. The following properties are then deduced: $l(x_1)+l(x_2)=l(x_1x_2)$. $l(x)$ is negative if $x<1$, zero if $x=1$ and positive if $x>1$, always increases as x increases, and its modulus becomes large without limit as $|x|$ becomes large or small without limit. It is (uniformly) continuous for all finite values of $x>0$, always increases as x increases, and has $\frac{1}{x}$ for differential coefficient. Hence, if any value whatever (y) is assigned to $l(x)$, there always exists one and only one (positive) value of x, and as y increases from $-\infty$ to ∞, x increases from 0 to ∞. The equation $y=l(x)$ may therefore be regarded as defining x as a function of y, say $E(y)$, which always increases with y, is never negative, is (uniformly) continuous, and becomes large or small numerically without limit as $|y|$ becomes large without limit in the positive or negative direction. The properties $E(0)=1$, $E(y_1+y_2)=E(y_1)E(y_2)$, $E(y)=[E(1)]^y\equiv e^y$ say, and $\dfrac{d}{dx}[E(x)]=E(x)$ are easily proved, and the result $x=e^y$ leads to $l(x)=\log_e(x)$. The number e is found $\left(\text{by transformation of } \underset{h=0}{\mathrm{Lt}}\dfrac{a^h-1}{h}\right)$ to be equal to $\underset{n=0}{\mathrm{Lt}}\,(1+n)^{\frac{1}{n}}$. From this result e is calculated to three decimal places.

The calculation of the expansions for the various elementary functions then follows on somewhat similar lines to those indicated by Professor Bromwich, the only notable difference being in the method used for the Binomial theorem.

The remainder of the course on real variables does not seem to require description here. The definite integral is of course treated fully, and it is pointed out that integration is not to be regarded as a process which may or may not be possible, but as an operation which may give rise to new functions, $\displaystyle\int_1^x \frac{dx}{x}$ and $\displaystyle\int_0^x \frac{dx}{1+x^2}$ being taken as examples.

As regards complex quantities, they are of course fully discussed in the first place, considerable practice being given in the use of the Argand diagram for such equations as $\omega=z^n$. The question of the definition of a function of a complex quantity is touched upon and the difficulty pointed out; the general definition is not given, it being suggested that the more advisable course is to examine first the properties of functions already known or immediately to be discovered. Then follow the differentiation of z^n and a^x (a being complex and x real), the latter leading as before to $\underset{h=0}{\mathrm{Lt}}\dfrac{a^h-1}{h}\equiv L(a)$. By means of the previous results the proof that

$$L(a)=l(|a|)+i\arg.a$$

is very simple, and properties of $L(a)$ corresponding to those of $l(a)$ are easily deduced, the most important being of course that if $\omega=L(z)$, a band of the ω plane corresponds to one revolution on the z plane, $L(z)$ being therefore many valued. If ω is confined to any one band of its plane, we may now write $z=E(\omega)$, where $E(\omega)$ repeats itself for different bands, and is therefore periodic. The properties of $E(\omega)$ are now deduced as before, including

$$E(u)=e^u,\ E(iv)=\cos v+i\sin v,$$

$$E(\omega_1+\omega_2)=E(\omega_1)E(\omega_2),\ E(u+iv)=e^u(\cos v+i\sin v),$$

$$\frac{d}{d\omega}\left\{E(\omega)\right\}=E(\omega) \text{ and } z^n=E[nL(z)].$$

The expansions for the various functions then follow on lines similar to those adopted for real variables.

As already indicated, I believe that some such course should precede any general discussion of infinite series, because it is at once simpler and more in harmony with modern developments than the old method of commencing with such series, whether preceded or followed by some account of their general theory. The latter proceeding appears to me calculated rather to crush and deaden the mind than to induce the true mathematical spirit, for the series are not introduced as the development of any general theory, and the student is forced on to them instead of being led up to their consideration. Moreover the nature of such series cannot be fully understood without considerable discussion of their properties, including the theories of uniform convergence and semi-convergent series, and these are certainly not fit subjects for a first course of analysis.

The above is the outline of a course which is intended both for students who wish to extend their mathematical education considerably beyond its limits and for those who intend to specialise in other subjects. Its success is essentially dependent on a large amount of numerical calculation and drawing being performed by the students, and on this understanding I hope that it may be of some use to the latter class as well as the former.

G. St. L. Carson.

REVIEW.

The Continuum as a Type of Order: an Exposition of the Modern Theory. With an Appendix on the Transfinite Numbers. By Edward V. Huntington. (Reprinted from the *Annals of Mathematics* for 1905; Cambridge, Mass., 50 cents.)

Mr. Huntington's first object is to give a systematic elementary account of the modern theory of the "continuum" regarded as a type of order.

Everyone who has had much to do with mathematics uses the conception of "the continuous independent variable," as represented geometrically by a point on a straight line (if the variable be real) or a plane (if it be complex). It was Dedekind and Cantor in 1872, and Cantor in 1882, who first gave us a clear and purely arithmetical account of the system of the real numbers and exactly what one means by saying: "the system of real numbers, or such and such a space, has *continuity*." And, in 1895, Cantor showed that the ordinal type (θ) of the series (0 ... 1) of real numbers is characterised by certain *purely ordinal* properties, in which no mention of, *e.g.*, "distance" occurs.

Russell has emphasised and developed the purely ordinal aspect. For him a "continuum" is a series of type θ, and he calls this "Cantor's second definition (1895) of continuity." I do not think Cantor ever called every such series a continuum, and, indeed, most mathematicians would call a series of some, but not all, of the real numbers in (0 ... 1) "discontinuous." Yet Harnack* has shown that such a series may be of type θ. However, Russell's theory is important to mathematics because many parts of function-theory can be shown to have a purely ordinal basis.†

Mr. Huntington explains lucidly, and by making use of much of Russell's work, what we mean by classes, simply-ordered classes or series, progressions (of type ω) and regressions (of type $^*\omega$), "dense" series (of type η), and "continuous" series (of type θ). This part of the work will be found useful, especially as Russell's contributions do not seem to be understood by everyone; while it seems to be a requisite in teaching function-theory by exacter methods to give some clear ideas of the ordinal properties of the number-system. An attempt (not quite, I think, successful) has been made in this direction by Harkness and Morley in their *Introduction to the Theory of Analytic Functions*.

In an Appendix, Mr. Huntington gives an account of the theory of well-ordered series and Cantor's transfinite numbers. Since much of the latter theory is still in the controversial stage (the laboratory stage of mathematics), an attempt at a connected account can, as yet, hardly be satisfactory. I will only mention one point. Mr. Huntington frequently refers to a paper by Dr. E. W. Hobson, which has served the useful purpose of directing attention to a point which, however, Dr. Hobson did not clearly see. This point is the question of the truth of a certain axiom; and Dr. Hobson would seem to doubt this truth, an opinion in which Mr. Huntington apparently seconds him. There is no reason why this alternative should not be adopted, but it must be observed that the conception of the product of an infinity of cardinal numbers (for example) involves this axiom.

This example shows the difficulties of giving an adequate account of a theory whose consequences are not yet at all fully realised.

PHILIP E. B. JOURDAIN.

MATHEMATICAL NOTES.

(Mr. Jourdain's Note on p. 327 should be numbered **201.** [D. 2. a.].)

202. [D. 2. a.]. Mr. Jourdain does not convince me. The order of ideas of which he would enforce the adoption is orthodox and adequate. Can he prove the contrary with regard to the order which starts from Dedekind's "Schnitt." In Mr. Jourdain's reference (end of § 6) I find that

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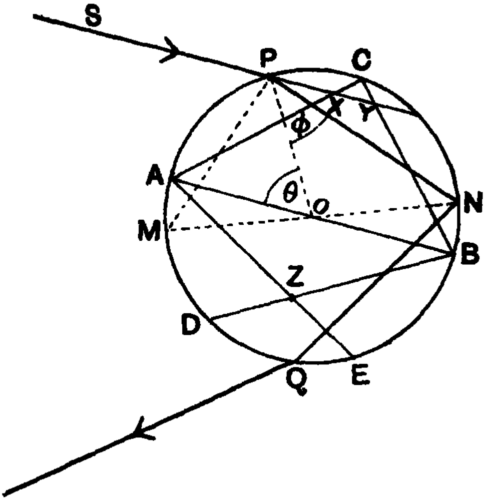
* *Math. Ann.*, Bd. 23 (1884).

† See Russell's *Principles*, pp. 296, 326; and part 2 of my paper in *Crelle's Journal*, Bd. 128 (1905).

Tannery before me has succeeded in annexing to it the Cantor-Weierstrass principle. Granted the "Schnitt," Mr. Jourdain's line 5 is irrelevant.

E. B. E.

203. [**T. a.**] The following is a simple construction for the paths of a series of parallel rays of light of given refrangibility through a uniform transparent sphere, with one internal reflection; as, *e.g.*, the solar rays through a raindrop, in the theory of the Rainbow, in Geometrical Optics.



Suppose an incident ray SP to be in the plane of the paper parallel to the diameter AOB of the sphere, O being its centre; and suppose that the straight line SP produced meets the chords AC, BC in X, Y respectively; C being taken so that $BC : BO$ in the ratio of the index of refraction. Let D, E be the points of trisection of arc AB; and let AE, BD intersect in Z. Then

(1) If M be the point diametrically opposite to N, the point of reflection of the ray, chord $PM = AX$.

(2) If the ray in question be that which suffers minimum deviation in the course of its passage through the drop, $BY = BZ$.

These results are easily proved. (1) enables us readily to construct the paths of a number of parallel rays of given refrangibility, the course of a ray after reflection being symmetrical with that before. Here C is fixed while the position of P varies. By (2), in like manner, we can trace the paths of those rays of various refrangibilities, which undergo minimum deviation, showing (approximately) the directions in which the various colours will be seen in their greatest intensity. Here the position of P varies with that of C. A series of such figures shows more clearly than any lengthy explanation, the way in which the rays passing through different drops give to an eye in a certain position the appearance of the rainbow. The result will be more obvious if the range of values of the index of refraction is somewhat extended.

Proof.

We have by construction $\mu = 2 \sin BAC = 2 \cos ABC$;

$$i.e. \ \sin\theta = 2\sin\phi \sin BAC = 2 \sin\phi \cos ABC,$$

where θ, ϕ are the angles of incidence and refraction. Thus we have

1. AX, which $=\dfrac{AO\sin\theta}{\sin BAC}$, $=AB\sin\phi$; *i.e.* $\sin\phi=\dfrac{AX}{AB}=\dfrac{PM}{MN}=\sin MNP$;
And supposing PM taken $=AX$ and MN the diameter through M, $\angle MNP=\angle NPO$, which is therefore the proper angle of refraction.

2. The total deviation $=\pi-2x$, where $x=\angle MOA=\angle MOP-\angle AOP=2\phi-\theta$.

Thus, substituting for ϕ above, we have $\sin\theta=2\sin\dfrac{\theta+x}{2}\cos ABC$, and for a minimum deviation

$$\cos\theta=\cos\frac{\theta+x}{2}\cos ABC.$$

Hence $\cos^2 ABC=\cos^2\theta+\frac{1}{4}\sin^2\theta$; or $\sin ABC=\dfrac{\sqrt{3}}{2}\sin\theta$.

Thus BY, which $=\dfrac{AO\sin\theta}{\sin ABC}$, $=\dfrac{AB}{\sqrt{3}}=BZ$, when the deviation is a minimum.

P. J. HEAWOOD.

204. [D. 2. a.] *Elementary Illustration of the Properties of Infinite Series.*

The writer has not met with a detailed discussion on these lines of the series mentioned below and thinks it may be of use to others who find that students have considerable difficulty in grasping the general theory.

1. Consider the family of curves

$$y=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots-\frac{(-x)^n}{n}\equiv S_n \text{ say,}$$

for which
$$\frac{dy}{dx}=\frac{1-(-x)^{n+1}}{1+x}.$$

It is easy to prove that if $x\leqq 1$ all the curves for which $n>p+1$ lie between the pth and $(p+1)$th curve and that the difference of the ordinates of these latter curves can be made small without limit by unlimited increase of p. Further, if $x<1$ the curves all tend to the common gradient $\dfrac{1}{1+x}$ as n increases without limit, but if $x=1$ the gradient is zero if n is even, and unity if n is odd, so that as they cross the line $x=1$ they part into two rapidly diverging families and there is a finite change of gradient (*not ordinate*) as x increases to unity. This latter change occurs more and more suddenly without limit as n increases indefinitely.

2. Next consider the family

$$y=x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^n}{n}\equiv T_n \text{ say,}$$

for which
$$\frac{dy}{dx}=\frac{1-x^n}{1-x}.$$

If $x<1$, all the curves after the pth lie within a limited distance above it, which distance can be made small without limit by unlimited increase of n, and as n increases indefinitely they tend to the common gradient $\dfrac{1}{1-x}$. As x increases to unity the gradient tends to the limit n and when $x=1$ the distance of the nth curve above the pth can be made large without limit by indefinite increase of n. Hence, for increasing values of n, the curves diverge more and more rapidly from Ox, *but an upper limit can be assigned to their ordinates so long as* $x<1$, this limit of course increasing indefinitely as x approaches unity.

3. Similarly for the family

$$y=\frac{x}{1^2}+\frac{x^2}{2^2}+\ldots+\frac{x^n}{n^2}\equiv U_n \text{ say,}$$

$$\frac{dy}{dx}=1+\frac{x}{2}+\ldots+\frac{x^n}{n}\equiv T_n$$

if $x \leqq 1$ all the curves after the pth lie within a limited distance above it, which distance can be made small without limit by unlimited increase of n. If $x<1$ their gradients tend to the same finite limit (since T_n does so), but if $x=1$ their gradients increase without limit. Hence as n increases without limit the curves approach one another more and more closely without limit and their ordinates tend to a definite finite value up to $x=1$ *inclusive*, but they cross the line $x=1$ with indefinitely increasing gradients.

4. Finally take the family

$$y=x+\frac{x^3}{3}-\frac{x^2}{2}+\frac{x^5}{5}+\frac{x^7}{7}-\frac{x^4}{4}+\ldots\equiv V_n \text{ say,}$$

$$\frac{dy}{dx}=\frac{1-x^{4m-1}}{1-x^2}-x\frac{1-x^{2m-1}}{1-x^2} \quad (n=3m)$$

$$=\frac{1-x^{4m-1}}{1-x^2}-x\frac{1-x^{2m-1}}{1-x^2}+x^{4m} \quad (n=3m+1)$$

$$=\frac{1-x^{4m-1}}{1-x^2}-x\frac{1-x^{2m-1}}{1-x^2}+x^{4m}+x^{4m+2}. \quad (n=3m+2)$$

If $x \leqq 1$ (the two cases now require separate treatment) all the curves after the pth lie within a limited distance of the pth, which distance can be made small without limit by unlimited increase of n. If $x<1$ their gradients tend to the limit $\frac{1}{1+x}$, but when $x=1$ they tend to the limits m, $m+1$, $m+2$ in the three cases above. Hence this case differs from those above in that the *ordinates* are limited up to $x=1$ *inclusive*, while the *gradients* increase sharply as x approaches unity. For large values of n the curves approach the line $x=1$ at a gradient nearly equal to $\frac{1}{2}$ and then turn sharply upwards with a large gradient, crossing $x=1$ with ordinates which do *not* tend to a limit near those for which x is slightly <1.

Further, it is easy to prove that

$$\tfrac{1}{6}x^{6n+1}+\tfrac{1}{4}x^{4n+2}>V_{6n}-S_{6n}>\tfrac{1}{8}x^{8n-1}+\tfrac{1}{6}x^{6n-1},$$

so that if $x<1$ V_{6n} and S_{6n} tend to the same limiting value for unlimited increase of n, but if $x=1$

$$\tfrac{5}{12}>V_{6n}-S_{6n}>\tfrac{7}{24},\text{*}$$

and in this case the series converge to different values. This illustrates the discontinuity which may occur in the sum of a series and the change in value caused by deranging the terms of a semi-convergent series.

For teaching purposes the argument should of course be illustrated by actual drawing.

G. St. L. Carson.

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* $\frac{5}{12}>\frac{1}{2}\log 2>\frac{7}{24}$ as should of course be the case.

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SOME USEFUL GROUPS IN THE TEACHING OF ELEMENTARY TRIGONOMETRY.

The operation (c) of taking the complement of an angle (a) is of order (period) two, since $90^\circ-(90^\circ-a)=a$. Similarly, the operation (s) of taking the supplement of an angle is of order two. If we perform these two operations in succession there results an operation (cs), which is of order four since it changes a into $a+90^\circ$. Hence the group $\{c, s\}$ generated by c and s is the dihedral rotation group of order 8. This fact follows directly from the general theorem, *Two operations of order two generate the dihedral rotation group whose order is twice the order of their product.** For the sake of simplicity we give the following independent proof of this fact:

Using rectangular co-ordinates and measuring angles in the ordinary manner, it is evident that the operation c and s are equivalent to the reflections of the terminal line of a on the bisector of the first quadrant and the y-axis respectively. These operations transform into itself the square whose centre is at the origin and whose sides are parallel to the axes. Hence this square is transformed into itself by all the operations which may be obtained by combining c and s in every possible way. That is, it is transformed into itself by $\{c, s\}$. Since it is known (and can readily be verified) that the group of movements of the square is the dihedral rotation group of order 8, it follows that $\{c, s\}$ is contained in this group. It could not be of order 4 since c and s are non-commutative. As the order of a subgroup divides the order of the group, $\{c, s\}$ must be identical with the group of movements of the square.

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* *Bulletin of the American Mathematical Society*, vol. 7 (1901), p. 424.

This group is of great historic interest since it is the one which enables us to reduce the solution of the general equation of the fourth degree to that of a cubic. In trigonometry it is perhaps most prominent as the complement-supplement group, while in arithmetic it appears as the group of the operations of subtracting from 2 and dividing by 2.* In elementary geometry it appears as the group of movements of the square, as was noted above. It is commonly called the octic group even if it does not have much greater claims for usefulness than several other groups of the same order—especially the quaternion group. We proceed to consider some fundamental uses of the octic group in elementary trigonometry.

The operations of this group may be written as follows:

$$\begin{array}{cccc} 1 & cs & cscs & sc \\ c & s & scs & csc. \end{array}$$

If these operations are applied to a the corresponding angles are

$$\begin{array}{cccc} a & a+90 & a+180^\circ & a-90 \\ 90^\circ-a & 180^\circ-a & 270^\circ-a & -a. \end{array}$$

It is very interesting to observe that these are the eight angles whose trigonometric functions are expressed in terms of functions of a in the early part of the text-books on elementary trigonometry. The trigonometric functions of all of these angles can be directly obtained from those of any two which correspond to generators of the octic group. For instance, if we know the trigonometric functions of the complement and the supplement of an angle, those of the other angles can be directly obtained by means of the given group symbols. We shall do this for sine and cosine†

$$\sin(a+90^\circ)=\sin csa=\sin ca=\cos a$$

$$\cos(a+90^\circ)=\cos csa=-\cos ca=-\sin a$$

$$\sin(a+180^\circ)=\sin cs\,.\,csa=\cos csa=-\sin a$$

$$\cos(a+180^\circ)=\cos cs\,.\,csa=-\sin csa=-\cos a$$

$$\sin(a-90^\circ)=\sin sca=\cos sa=-\cos a$$

$$\cos(a-90^\circ)=\cos sca=\sin sa=\sin a$$

$$\sin(270^\circ-a)=\sin sc\,.\,sa=\sin sca=-\cos a$$

$$\cos(270^\circ-a)=\cos sc\,.\,sa=-\cos sca=-\sin a$$

$$\sin(-a)=\sin cs\,.\,ca=\cos csa=-\sin a$$

$$\cos(-a)=\cos cs\,.\,ca=\sin csa=\cos a.$$

If we represent the operation of increasing an angle by 90° by p and suppose the functions of $a+90^\circ$ as well as those of

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* *Quarterly Journal of Mathematics*, vol. 37 (1905), p. 82.

† Cf. *Quarterly Journal of Mathematics*, vol 37, (1906), p. 228.

$90^\circ - a$ to be known, we may readily find those of the others in the following manner:

$$\begin{aligned}
\sin(a+180^\circ) &= \sin p^2a = \cos pa = -\sin a \\
\cos(a+180^\circ) &= \cos p^2a = -\sin pa = -\cos a \\
\sin(a-90^\circ) &= \sin p^3a = -\sin pa = -\cos a \\
\cos(a-90^\circ) &= \cos p^3a = -\cos pa = \sin a \\
\sin(180^\circ - a) &= \sin cpa = \cos ca = \sin a \\
\cos(180^\circ - a) &= \cos cpa = -\sin ca = -\cos a \\
\sin(270^\circ - a) &= \sin cp^2a = -\sin ca = -\cos a \\
\cos(270^\circ - a) &= \cos cp^2a = -\cos ca = -\sin a \\
\sin(-a) &= \sin cp^3a = -\cos ca = -\sin a \\
\cos(-a) &= \cos cp^3a = \sin ca = \cos a.
\end{aligned}$$

The advantages of the present point of view result from the fact that it associates the group and these eight angles in such a way that a knowledge of the one throws light on the other. If we know the octic group, we know directly what two angles may be made fundamental in the study of the others.* The necessary and sufficient condition is that the two angles correspond to generators of the octic group. In other words, they must correspond to two non-commutative operations of this group, since any two non-commutative operations of a non-abelian group of order p^3, p being a prime number, generate the group and any two generators are non-commutative.

The properties of the octic group suggest a large number of exercises with respect to the eight angles under consideration. For instance, the functions of all of them may be expressed in terms of the functions of any other one of them just as easily as in terms of the functions of a. Since the octic group has only one invariant operation besides the identity the corresponding angle cannot be used as one of the two fundamental angles, while each of the non-invariant operations can be used as one of these two angles. If a is the angle corresponding to the identity, as above, $a+180^\circ$ corresponds to the invariant operation of order 2.

The use of the octic group makes the work entirely analytic after the trigonometric functions of the fundamental angles are known. The main interest, however, lies in the connections which it exhibits and the comprehensive view which it affords. The teacher should be familiar with this point of view even if he does not consider it wise to present it to the beginner.

Instead of considering the operations of subtracting from 90° and 180° we might consider those of subtracting from any other angles.[*] For instance, if t represents the operation of subtracting from 45° while c has the same meaning as before, the operation tc is of order 8, and hence $\{c, t\}$ is the dihedral rotation group of order 16. The operations of this group and the corresponding angles may be written in the same manner as those of the octic group. In fact, it is easy to see that $\{c, t\}$ includes $\{c, s\}$, so that the results obtained above will be appear as a special case under this larger group. We proceed to give a few illustrative examples.

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* We could not use only one fundamental angle since the octic group is non-cyclic.

$$\sin(a+45^\circ)=\sin tca=\cos ta=\tfrac{1}{2}\sqrt{2}(\cos a+\sin a)$$ [†]

$$\cos(a+45^\circ)=\cos tca=\sin ta=\tfrac{1}{2}\sqrt{2}(\cos a-\sin a)$$

$$\begin{aligned}\sin(a+90^\circ)&=\sin(tc)^2a=\tfrac{1}{2}\sqrt{2}(\cos tca+\sin tca)\\&=\tfrac{1}{2}(\cos a-\sin a+\cos a+\sin a)=\cos a\end{aligned}$$

$$\begin{aligned}\cos(a+90^\circ)&=\cos(tc)^2a=\tfrac{1}{2}\sqrt{2}(\cos tca-\sin tca)\\&=\tfrac{1}{2}(\cos a-\sin a-\cos a-\sin a)=-\sin a\end{aligned}$$

$$\begin{aligned}\sin(a+180^\circ)&=\sin(tc)^4a=\cos(tc)^2a=-\sin a\\\cos(a+180^\circ)&=\cos(tc)^4a=-\sin(tc)^2a=-\cos a\end{aligned}$$

$$\begin{aligned}\sin(a-90^\circ)&=\sin(tc)^6a=-\sin(tc)^2a=-\cos a\\\cos(a-90^\circ)&=\cos(tc)^6a=-\cos(tc)^2a=\sin a\end{aligned}$$

$$\begin{aligned}\sin(-a)&=\sin(tc)^2ca=\cos(tc)^2a=-\sin a\\\cos(-a)&=\cos(tc)^2ca=\sin(tc)^2a=\cos a\end{aligned}$$

$$\begin{aligned}\sin(a+135^\circ)&=\sin(tc)^3a=\cos tca=\tfrac{1}{2}\sqrt{2}(\cos a-\sin a)\\\cos(a+135^\circ)&=\cos(tc)^3a=-\sin tca=-\tfrac{1}{2}\sqrt{2}(\cos a+\sin a).\end{aligned}$$

These examples seem sufficient to illustrate the method of work. The general theory of these subtraction groups is given in the articles published in the *Annals of Mathematics* to which reference is made above. The object of the method is to exhibit relations which are apt to be overlooked. It aims to clarify and intensify the knowledge of fundamental facts rather than to extend this knowledge. Some of the relations between the functions of the same angle afford another very interesting application of the group concept. The two operations of subtracting from unity and dividing unity may be represented by s_1, s_2 respectively. It is evident that each of these operations is of order 2 and that their product s_1s_2 is of order 3. Hence they generate the dihedral rotation group of order 6. This is well illustrated by the six values of an anharmonic ratio and the group of movements of the equilateral triangle.

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[*] *Cf. Annals of Mathematics*, vol. 6 (1905), p. 41.

[†] Since t is one of the assumed generators of our group of order 16 the functions of the angle (45° − a) which corresponds to t are supposed to be known.

The six operations of $\{s_1, s_2\}$ may be represented as follows:

$$\begin{matrix} 1 & s_1s_2 & s_2s_1 \\ s_1 & s_2 & s_1s_2s_1. \end{matrix}$$

If we perform these operations on $\sin^2\alpha$ the resulting functions are, in order,

$$\begin{matrix} \sin^2\alpha & \sec^2\alpha & -\cot^2\alpha \\ \cos^2\alpha & \csc^2\alpha & -\tan^2\alpha. \end{matrix}$$

Since the six operations in question constitute a group the same six functions are obtained if the operations are performed on any one of them. This method exhibits not only the close relations between these functions and the values of an anharmonic ratio, but it may also be regarded as a special case of a group of subtraction and division. The fundamental properties of these groups have been developed in an article published in volume 37 of the *Quarterly Journal of Mathematics*. The same volume contains other interesting examples of the application of group theory to trigonometry. G. A. MILLER.

UNIVERSITY OF ILLINOIS.

ON HIGHER TRIGONOMETRY.

III. THE ELEMENTARY TRANSCENDENTAL FUNCTIONS.

Algebraic Lemma.

The following system of algebraic inequalities is of great value and of wide application:

$$\text{(i)}\ 1-a<\frac{1}{1+a},\quad 1+a<\frac{1}{1-a}$$

for all positive proper fractional values of a;

$$\text{(ii)}\ \prod_{r=1}^{r=n}(1-a_r)>1-\Sigma_1+\Sigma_2-\ldots-\Sigma_{2k-1}$$

$$\text{and}\quad <1-\Sigma_1+\Sigma_2-\ldots+\Sigma_{2k}$$

for positive proper fractional values of $a_1, a_2, \ldots a_n$, and values of k such that $2k<n$; Σ_r denoting the sum of the r-products of the a's.

These inequalities (ii) may be given otherwise in the form

$$\Pi(1-a)=1-\Sigma_1+\Sigma_2\ldots+(-)^{k-1}.\Sigma_{k-1}+(-)^k.\phi_k.\Sigma_k,$$

where ϕ_k is a function of $a_1, a_2, \ldots a_n$, whose value is a positive proper fraction when positive proper fractional values are given to the a's. In particular

$$1-na<(1-a)^n<1-na+\frac{n(n-1)}{1.2}a^2,$$

for positive proper fractional values of a, and positive integral values of n.

The inequalities $1 > \Pi(1-a) > 1-\Sigma_1$ have been freely used, *e.g.* in the theory of infinite products, but it does not seem to have been noted that they admit of the above useful extension.

The Exponential Function.

The function of z and n which is expressed by $(1+z/n)^n$ can be expanded for all values of z and for positive integral values of n, and this enables us to prove that it tends to a limit (a function of z alone) as n tends to infinity by integral variation. This function of z can be shown to have simple and interesting properties which make it fundamental to the higher analysis, and on account of which it is designated the exponential function:

$$(1+z/n)^n = 1+z+p_1 \cdot \frac{z^2}{2!}+p_2 \cdot \frac{z^3}{3!}+\ldots+p_{n-1} \cdot \frac{z^n}{n!},$$

where p_r denotes the product $(1-1/n)(1-2/n)\ldots(1-r/n)$; it is clear from the Lemma that $1 > p_r > 1-r(r+1)/2n$ for all values of r considered, and therefore, if we write $p_r = 1-\epsilon_r$, ϵ_r tends to zero as n tends to infinity, *for finite values of* r.

Hence, k having a positive integral value, we may write

$$(1+z/n)^n = E(z, k) - \eta(z, k, n)$$

$$+p_{k-1} \cdot \frac{z^k}{k!}\left\{1+\left(1-\frac{k}{n}\right)\frac{z}{k+1}+\left(1-\frac{k}{n}\right)\left(1-\frac{k+1}{n}\right)\frac{z^2}{(k+1)(k+2)}+\ldots\right\}$$

where $E(z, k) \equiv 1+z+\frac{z^2}{2!}+\ldots+\frac{z^{k-1}}{(k-1)!}$ and is independent of n,

$$\eta(z, k, n) \equiv \epsilon_1 \cdot \frac{z^2}{2!}+\epsilon_2 \cdot \frac{z^3}{3!}+\ldots+\epsilon_{k-2} \cdot \frac{z^{k-1}}{(k-1)!},$$

and tends to zero when n tends to infinity, *for finite values of* k *and* z.

Thus $|(1+z/n)^n - E(z, k)| < |\eta(z, k, n)| + |z|^k/(k-|z|).(k-1)!$,

provided that $k > |z|$, by the ordinary method of applying a geometric approximation to obtain an upper limit to the modulus of the remainder.

Now the function $E(z, k)$ can be shown by Limit-Theorem IV. above [again applying the geometric series approximation] to tend to a limit, $E(z)$, as k tends to infinity, for all finite values of z; *this is generally written*

$$E(z) = 1+z+\frac{z^2}{2!}+\ldots+\frac{z^r}{r!}+\ldots \textit{ ad inf.}$$

the "infinite series" on the right being "convergent" for all finite values of z.*

And we can now write

$$|(1+z/n)^n-E(z)|<|\delta(z,k)|+|\eta(z,k,n)|+\gamma(z,k),$$

where $\delta(z,k)$ is a function which tends to zero as k tends to infinity for all finite values of z; and $\gamma(z,k)=|z|^k/(k-|z|)(k-1)!$

Then, corresponding to any assigned finite positive real number, ϵ, *however small*, we can choose

(i) a value of k so great that $|\delta(z,k)|<\epsilon/3$ and that $\gamma(z,k)<\epsilon/3$.

(ii) a value of n so much greater that $|\eta(z,k,n)|<\epsilon/3$,

i.e. a value (N) of n can be found such that

$$|(1+z/n)^n-E(z)|<\epsilon, \text{ provided } n>N,$$

and therefore $(1+z/n)^n$ *tends to the limit* E(z) *as* n *tends to infinity by integral variation, for all finite values of* z.

[In the particular cases given by the values $+1$ and -1 of z the student may obtain exercise in the new ideas by going over the proof and applying the more obvious Limit-Theorems II. and III.; establishing

(i) that $\text{Lt}(1+1/n)^n$ is equal to the limiting value of the increasing real function $(1+1+1/2!+\ldots+1/n)$;

(ii) that $\text{Lt}(1-1/n)^n$ is equal to the common limiting value of the two real functions

$$\left(\frac{1}{2!}-\frac{1}{3!}+\ldots-\frac{1}{(2n-1)!}\right),\ \left(\frac{1}{2!}-\frac{1}{3!}+\ldots+\frac{1}{2n!}\right),$$

the former increasing, the latter decreasing. This case arises out of the inequalities connected with $(1-1/n)^n$; the course of higher trigonometry may satisfactorily commence with it and the student be very valuably convinced that $(1-1/n)^n$ is less than $\frac{1}{2}$ for all positive integral values of n, greater than $\frac{1}{3}$ for integral values greater than 5. Compare also Professor Bromwich's remarks in the *Gazette*, vol. iii., p. 87.]

The limiting values of $(1+1/n)^n$ and $(1-1/n)^{-n}$ are easily shown to be equal, for

$$(1-1/n)^{-n}=\frac{n}{n-1}\cdot\left(1+\frac{1}{n-1}\right)^{n-1};$$

and their common value, an irrational number [cf. Chrystal, chap. xxviii., § 3], is denoted by e.

It is next necessary to establish in some way that the function $(1+1/x)^x$ tends to the value e when x tends to infinity *by*

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* The student need not up to this stage have given any attention to the theory of infinite series; he will probably have had his attention drawn to the geometric infinite series and its application to recurring decimals.

continuous real variation. This is equivalent to the statement that the function $(1+x)^{1/x}$ is continuous through the value zero of the argument. Now, we know that this function has one and only one positive real value for each real value of x greater than -1, *the ideas of continuity being applied to justify this statement with reference to irrational values of* x [in the definition on the Dedekind-scheme of a power when the index is irrational]; also we know that it tends to the value e when x tends to zero by the reciprocals of the integers. The *assumption* that $(1+x)^{1/x}$ is continuous through the value zero of the argument can, therefore, hardly be regarded as violent.*

If then we write ξx for n in the function $(1+x/n)^n$ and suppose n to have positive integral values, ξ and x to have real values,

$$(1+x/n)^n=(1+1/\xi)^{\xi x}=\{(1+1/\xi)^{\xi}\}^x,$$

and therefore the limit to which it tends is e^x, *i.e.* $E(x)=e^x$.

Hence the function $E(z)$, defined by the convergent infinite series

$$1+z+\frac{z^2}{2!}+\ldots+\frac{z^r}{r!}+\ldots \textit{ ad inf.},$$

is called the exponential function, having the addition-theorem $E(x_1)\times E(x_2)=E(x_1+x_2)$ *for real values of* x_1 *and* x_2.

Derivative of e^x^.

The inequalities

$$1-x<(1-x/n)^n<1-x+(1-1/n)\frac{x^2}{2}<1-x+\frac{x^2}{2},$$

which hold for positive real values of x if n has an integral value greater than x, give as limiting case the inequalities

$$1-x<e^{-x}<1-x+\frac{x^2}{2}, \textit{ i.e. } x>1-e^{-x}>x-\frac{x^2}{2}$$

for positive real values of x; and therefore

$$\operatorname*{Lt}_{h=+0} e^{x+h}\left(\frac{1-e^{-h}}{h}\right), \textit{ i.e. } \frac{d}{dx}\cdot e^x, =e^x,$$

a very convincing and elementary proof of this fundamental fact.

Powers with Complex Indices.

It remains to exhibit how this theory gives a satisfactory means of defining a power for an index which is a complex number.

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* If need is felt for further satisfaction, reference can be made to Chrystal, chap. xxv., § 13, where it is proved that the function $(1+1/x)^x$ increases steadily when an increasing system of positive *rational* values is given to x. Also to Harnack's *Introduction to the Calculus*, Bk. I., ch. vi., § 28, or to Gibson's *Calculus*, ch. v., § 48. *But some assumption similar in type to that suggested above must always be implicitly made*; cp. footnote p. 332, *Gazette*, July, 1906.

For real values of x_1 and x_2, we know that

$$(1+x_1/n)^n(1+x_2/n)^n \text{ and } \{1+(x_1+x_2)/n\}^n$$

each tend to the limit $e^{x_1+x_2}$ when n tends to infinity by positive integral variation. Hence

$$\{1+(x_1+x_2)/n+x_1x_2/n^2\}^n-\{1+(x_1+x_2)/n\}^n$$

tends to the limit zero for real values of x_1 and x_2.

Now
$$|\{1+(z_1+z_2)/n+z_1z_2/n^2\}^n-\{1+(z_1+z_2)/n\}^n|$$
$$=|\sum_{r=1}^{r=n} {}_nC_r . \{1+(z_1+z_2)/n\}^{n-r}.(z_1z_2/n^2)^r|$$
$$<\sum_{r=1}^{r=n} {}_nC_r\{1+(\rho_1+\rho_2)/n\}^{n-r}.(\rho_1\rho_2/n^2)^r,^*$$

clearly, if ρ_1, ρ_2 denote $|z_1|$, $|z_2|$;

$$\textit{i.e.} < \{1+(\rho_1+\rho_2)/n+\rho_1\rho_2/n^2\}^n-\{1+(\rho_1+\rho_2)/n\}^n$$

and therefore tends to zero when n tends to infinity.

Hence the two functions

$$(1+z_1/n)^n(1+z_2/n)^n \text{ and } \{1+(z_1+z_2)/n\}^n$$

tend to the same limiting value; *i.e.* the function $E(z)$ has the addition-theorem

$$E(z_1)\times E(z_2)=E(z_1+z_2)$$

for all values of z_1, z_2 real or complex.

In particular $E(x+\iota y)=e^x.E(\iota y)$, for real values of x and y; and the complex function $E(\iota x)$, which is the limit of $(1+\iota x/n)^n$, may be suitably used to define the symbol $e^{\iota x}$ for real values of x.

The Trigonometric Series.

The real components of $E(\iota x)$ are given by the two convergent infinite series

$$C(x)\equiv 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\dots \textit{ ad inf.},$$

$$S(x)\equiv x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots \textit{ ad inf.}$$

And the fact that $\cos^r(x/n)$ and $\tan^r(x/n)/(x/n)^r$ tend to the value $+1$ when n tends to infinity, for all integral values of r from 1 to n, enables us to show that $e^{\iota x}=\cos x+\iota\sin x$: for

$$(\cos x/n+\iota\sin x/n)^n=\cos^n(x/n)\left\{1+\frac{\iota x}{n}\cdot\frac{\tan(x/n)}{x/n}\right\}^n;$$

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[*] The device used here looks as if it might have a wide application; it is again applied to the Infinite Products. See p. 362.

and, therefore, if we write $1+\epsilon_r$ for $\tan^r(x/n)/(x/n)^r$, so that ϵ_r is positive and tends to the limit zero when n tends to infinity,

$$\cos x+\iota\sin x=\cos^n(x/n)\left\{(1+\iota x/n)^n+\sum_{r=1}^{r=n}{}_nC_r\,.\,\epsilon_r\,.\left(\frac{\iota x}{n}\right)^r\right\};$$

and if η denotes the greatest of the ϵ's,

$$\left|\Sigma_n C_r\,.\,\epsilon_r\,.\left(\frac{\iota x}{n}\right)^r\right|<\eta\{(1+x/n)^n-1\}.$$

Now when n tends to infinity, $\cos^n(x/n)$ tends to the value $+1$, $(1+\iota x/n)^n$ tends to the value $e^{\iota x}$, $(1+x/n)^n$ to the value e^x and η to the value zero; therefore, $\cos x+\iota\sin x\equiv e^{\iota x}$ for real values of x.

Thus $\cos x=C(x)$, $\sin x=S(x)$ giving the infinite expansions for the sine and the cosine.

The Logarithmic Series.

Integration of the algebraic identity

$$(1+z)^{-1}\equiv 1-z+z^2-\ldots+(-)^{r-1}.z^{r-1}+(-)^r.z^r(1+z)^{-1}$$

leads to a proof of the fact that $\log_e(1+z)$ is the limit-function denoted by

$$z-z^2/2+\ldots+(-)^{r-1}.z^r/r+\ldots \textit{ ad inf.},$$

provided that $|z|<1$. [Cp. Gibson's *Introduction to the Calculus*, p. 165, Ex. 5.]

[The fact that this method provides so straightforward a rigorous demonstration of the logarithmic theorem is another link in the argument for taking the Calculus as the first branch of Higher Mathematics, making the Definite Integral our first application of the Limit-theorems given above.]

At this stage a short course on the general theory of Infinite Series would be introduced with advantage, before passing on to the other simple type of Infinite Sequence which appears naturally (in a manner analogous to that in which the Infinite Series have come) through

The Infinite Trigonometric Products.

Taking the generalised definitions of the sine- and cosine-functions to be expressed by the formulae

$$2\iota\sin z=E(\iota z)-E(-\iota z),\quad 2\cos z=E(\iota z)+E(-\iota z),$$

these functions have to do with limit-functions determined by the sequences

$$(1+\iota z/n)^n\mp(1-\iota z/n)^n.$$

Now it follows from the solution of $z^n=1$, that

$$(1+\iota z/n)^n-(1-\iota z/n)^n = 2\iota z \prod_{r=1}^{r=n-1}\left\{1-\frac{z}{n\tan(r\pi/n)}\right\}$$
$$= 2\iota z \prod_{r=1}^{r=\nu}\left\{1-\frac{z^2}{n^2\tan^2(r\pi/n)}\right\},$$

where ν denotes $(n-1)/2$ or $(n-2)/2$, according as n is odd or even.

The fact that, throughout the product, $r < n/2$ provides that $n\tan(r\pi/n) > r\pi$; and, *for finite values of* r, $n\tan(r\pi/n)$ tends to $r\pi$ as a limit when n tends to infinity: thus, we may write

$$\prod_1^r\left\{1-\frac{z^2}{n^2\tan^2(r\pi/n)}\right\} = \{P(z,k)+\eta(z,k,n)\}\,.\,R(z,k,n),$$

where

$$P(z,k)\equiv\prod_1^k(1-z^2/r^2\pi^2),\quad R(z,k,n)\equiv\prod_{k+1}^{\nu}\left\{1-\frac{z^2}{n^2\tan^2(r\pi/n)}\right\},$$

and $\eta(z,k,n)$ tends to zero when n tends to infinity, *for finite values of* k *and* z.

But if the expansion of $R(z,k,n)$ be denoted by

$$1-a_1\,.\,z^2+a_2\,.\,z^4-\ldots,$$

the a*'s denoting positive real constants,*

$$|1-R(z,k,n)| < a_1\,.\,\rho^2+a_2\,.\,\rho^4+a_3\,.\,\rho^6+\ldots$$
$$< \prod_{k+1}^{\nu}\left\{1+\frac{\rho^2}{n^2\tan^2(r\pi/n)}\right\}-1^*$$
$$< \prod_{k+1}^{\nu}(1+\rho^2/r^2\pi^2)-1,$$

which, provided that $k\pi > \rho$,

$$< \{1/\Pi(1-\rho^2/r^2\pi^2)\}-1,$$

and therefore $$< \pi^2/(\pi^2-\rho^2\,.\,\sum_{k+1}^{\nu}1/r^2)-1.$$ [See the inequalities, p. 357.]

Thus it follows, from the algebra of the convergent series $\sum^{\infty} 1/r^2$, that

$$R(z,k,n)=1+\gamma(z,k,n),$$

where $\gamma(z,k,n)$ is a function whose value is diminished indefinitely when a system of indefinitely increasing values is given to k, z having any finite value and n any integral value so great that $n > 2k$.

Also, an obvious slight variation of the above analysis proves that the "infinite product" $\prod_1^{\infty}(1-z^2/r^2\pi^2)$ is "convergent," *i.e.*

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* Cp. p. 360 of this number.

that the sequence, $\prod_1^n(1-z^2/r^2\pi^2)$, involved determines a limit-function, which may be denoted by $P(z)$; hence

$$P(z, k)=P(z)+\delta(z, k),$$

where $\delta(z, k)$ is a function which, for any finite value of z, tends to zero when k tends to infinity.

Thus, finally,

$$\prod_1^\nu\left\{1-\frac{z^2}{n^2\tan^2(r\pi/n)}\right\}=\{P(z)+\delta(z, k)+\eta(z, k, n)\}\{1+\gamma(z, k, n)\};$$

and it follows (cp. p. 359) that, corresponding to any assigned finite positive real number, ϵ, *however small*, a value (N) of n can be chosen (greater than a value of k chosen first), such that

$$\left|\prod_1^\nu\left\{1-\frac{z^2}{n^2\tan^2(r\pi/n)}\right\}-P(z)\right|<\epsilon, \text{ provided } n>N.$$

Hence

$$\begin{aligned}\sin z&=z\,.\,P(z)\\&=z\,.\prod_1^\infty(1-z^2/r^2\pi^2).\end{aligned}$$

Similarly, from the algebraic identity,

$$(1+\iota z/n)^n+(1-\iota z/n)^n\equiv 2\prod_1^\nu\left\{1-\frac{z^2}{n^2\tan^2(\overline{2r-1}\,.\,\pi/n)}\right\},$$

where ν denotes $(n-1)/2$ or $n/2$ according as n is odd or even, it follows by an exactly similar process that

$$\cos z=\prod_1^\infty\{1-4z^2/(2r-1)^2\pi^2\}.$$

And it is clear that the method might be applied to the fractional infinite series for cosec z, sec z, cot z, and tan z.*

The investigation given here seems to provide a serious criticism of Mr. Hardy's dogma, that the use of the function $(1+z/n)^n$ in the Exponential theory "is instructive and important, but essentially accessory and not fundamental, and to base the exponential theorem on it is *logically quite wrong*" [italics supplied]; and that the factor-theorem "is *really* difficult," providing "a *crux* in the teaching of trigonometry" [*Gazette*, p. 285].

That the line of argument is logical will hardly be denied, and that it gives valuable insight into the infinite sequence as an instrument will probably be admitted by anyone who cares to devote the best part of a term's teaching to it. The amount of detail is perhaps a little alarming, but there is only one dominating idea, which is not hard, for the whole theory. And the writer repeats that a knowledge of this theory is not for the immature schoolboy mind; very serious damage has been done to mathematical training by attempts to gloss over the inherent difficulties which lie at the foundations of the Higher Trigonometry.

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* For another method of establishing these results, see a paper by the writer of this article in the *Proc. Edin. Math. Soc.*, vol. xxii.; the treatment there is of the real case, but the analysis is suggestive in the present connection and instructive to the beginner.

D. K. PICKEN.

ON A PROBLEM IN MECHANICS AND THE NUMBER OF ITS SOLUTIONS.

To find the position of equilibrium of a particle which is acted on by forces of given magnitudes directed towards given points.

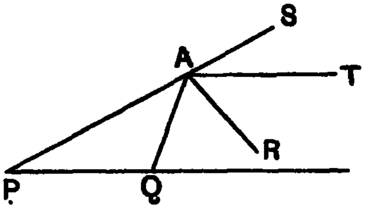
This problem was suggested by Mr. P. A. Hillhouse, and arose in the first instance from the consideration of forces acting in a crane. The following note does not offer a solution of the problem, but contains a proof by Mr. A. L. Jones that there is at most one position of equilibrium when the forces are all of one sign, *i.e.* either all tensions or all thrusts. This is followed by an investigation of the total number of solutions, real and imaginary, for all combinations of thrusts and tensions, when the magnitude only of each force, and not its sign, is given. This number is surprisingly large. It would of course be of more practical interest to find the number of *real* solutions for each particular combination of thrusts and tensions, but I have only succeeded in doing this when the given points are in a straight line, and in one or two other special cases. The results for these are stated without proof at the end of the note.

When the forces are all of one sign the single real position of equilibrium could be approximately found experimentally by means of threads, all knotted together at one end, which are passed through small smooth rings at the given points, and carry at their free ends weights proportional to the given forces. The position taken by the knot will then be the position of equilibrium.

Mr. Jones' proof (for forces all of one sign) depends on the simple lemma that *two equal positive forces in the directions PA, AQ have a resultant whose direction makes an acute angle with the direction of PQ.* For the resultant AR bisects the angle QAS, formed by AQ and PA produced; and the line through A in the direction PQ makes with AR an angle less than half QAS, that is, an acute angle. The resultant cannot vanish unless A is on the line PQ and external to PQ; and the lemma holds if A is internal to PQ.

Suppose that P is a point such that given positive forces

$t_1, t_2, \dots, t_n$ acting from P towards given points $A_1, A_2, \dots, A_n$ are in equilibrium, and that if possible a second point Q exists for which the same is true. Then the system of forces $t_1, t_2 \dots, t_n$ along $PA_1, PA_2, \dots, PA_n$, and the system of forces $t_1, t_2, \dots, t_n$ along $A_1Q, A_2Q, \dots, A_nQ$, are in equilibrium separately, and therefore in equilibrium jointly. But t_r along PA_r and t_r along A_rQ have a resultant which makes an acute angle with PQ; and the two systems are therefore equivalent to a single system of positive forces all of which make acute angles with PQ. This system cannot be in equilibrium unless all the forces vanish; and this cannot happen unless $A_1, A_2, \dots, A_n$ are collinear, in which case there is either no position of equilibrium or an infinite number of such positions. If the given points are not collinear there can only be one real position of equilibrium at most.



If forces $t_2, t_3, \dots, t_n$ along $A_1A_2, A_1A_3, \dots, A_1A_n$ have a resultant t_1, then A_1 is the position of equilibrium. The necessary and sufficient conditions that there may be a position of equilibrium other than $A_1, A_2, \dots, A_n$ are that the resultant of the forces $t_1, \dots, t_{r-1}, t_{r+1}, \dots t_n$ along $A_rA_1, \dots A_rA_{r-1}, A_rA_{r+1}, \dots A_rA_n$ should be greater than t_r, for all values of r from 1 to n. For if these n conditions hold, and the particle be placed at any one of the points $A_1, A_2, \dots, A_n$, it will be drawn away by the forces which act upon it; and the forces cannot send the particle away to infinity, if they are tensions. For tensions the position of equilibrium is stable, and for thrusts unstable.

A slight extension of the above proof shows that if at one of the given points, say at A_1, the resultant of the forces $t_2, t_3, \dots, t_n$ along $A_1A_2, A_1A_3, \dots, A_1A_n$ is equal to or less than t_1, then for any other point A_r the resultant of a like set of forces is greater than t_r. From this it follows that $n-1$ of the n conditions mentioned above hold of necessity.

We will now consider the problem analytically and find the total number of solutions, real and imaginary, when the absolute magnitudes only of the forces are given. We consider first the case in which the given points are in a plane. Let a_r, b_r be the rectangular coordinates of A_r $(r=1, 2, \dots n)$, and x, y the coordinates of the particle P, which is in equilibrium under the system of forces $t_1, t_2, \dots t_n$ along $PA_1, PA_2 \dots PA_n$. The equations for equilibrium, by resolving along the axes, are

$$\frac{t_1(x-a_1)}{r_1}+\frac{t_2(x-a_2)}{r_2}+\ldots+\frac{t_n(x-a_n)}{r_n}=0,$$

$$\frac{t_1(y-b_1)}{r_1}+\frac{t_2(y-b_2)}{r_2}+\ldots+\frac{t_n(y-b_n)}{r_n}=0,$$

where $$r^2_p=(x-a_p)^2+(y-b_p)^2.$$

Assuming $r_1, r_2, \ldots r_n$ to be always positive,* we cannot in general find the solutions of these equations without at the same time finding the solutions for all the 2^{n-1} combinations of thrusts and tensions among the given forces. We shall, however, assume that $r_1, r_2, \ldots r_n$ are capable of either sign, while $t_1, t_2, \ldots t_n$ remain invariably positive, as this gives us the same system of equations, and leaves the values of the unknowns $x, y, r_1, r_2, \ldots r_n$ unrestricted. In order to discuss the infinite (as well as the finite) solutions we introduce another unknown u so as to make the equations homogeneous. Then the system of equations for the ratios of $u, x, y, r_1, \ldots r_n$ is

$$\frac{t_1(x-a_1u)}{r_1}+\ldots+\frac{t_n(x-a_nu)}{r_n}=0, \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{t_1(y-b_1u)}{r_1}+\ldots+\frac{t_n(y-b_nu)}{r_n}=0, \quad\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$(x-a_1u)^2+(y-b_1u)^2=r_1^2, \quad\ldots\ldots\ldots\ldots\ldots(3)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$(x-a_nu)^2+(y-b_nu)^2=r_n^2. \quad\ldots\ldots\ldots(n+2)$$

Equations (1), (2) are written in a fractional form so as to be less cumbersome, but can be at once made integral by multiplying by $r_1r_2\ldots r_n$. Thus there are two equations of degree n, and n equations of degree 2, and the total number of solutions, taking account of infinite and multiple solutions, is n^22^n. There are, however, a large number of irrelevant solutions which do not satisfy the conditions of the problem; their number must be found and subtracted from n^22^n in order to find the true number of solutions of the problem.

In particular, all infinite solutions (those in which $u=0$) are irrelevant, and must be rejected; it will be found that there are only two different infinite solutions, but that each is multiple to a high degree. In addition some multiple and non-multiple finite solutions are also irrelevant. In general the relevant solutions are all non-multiple and finite.

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* The function $t_1r_1+t_2r_2+\ldots+t_nr_n$ is the potential of the system of forces, and has a stationary value at a position of equilibrium P. The curve of the family

$$t_1r_1+t_2r_2+\ldots+t_nr_n=\text{constant},$$

which passes through P, has a node at P; and a node is a position of equilibrium. If two positions of equilibrium move up to coincidence the node becomes a cusp.

The chief difficulty is to find the multiplicity of the multiple solutions. To do this we make use of a principle which is seldom applied: *In a given system of n non-homogeneous equations for n unknowns $x_1, \dots x_n$ the multiplicity of the solution*

$$x_1=x_2=\dots=x_n=0$$

is the product of the degrees of the terms of lowest degree in the equations, provided that these lowest terms when equated to zero have no proper solution, that is, no solution other than $x_1=x_2=\dots=x_n=0$. Again the multiplicity of the solution $x_1=a_1$, $x_2=a_2, \dots x_n=a_n$ is found by the same rule after substituting $x_1+a_1, \dots x_n+a_n$ for $x_1, \dots x_n$ in all the equations. Finally the rule is applied to a system of *homogeneous* equations by making them non-homogeneous, by putting a non-vanishing unknown, of which there must be at least one in any proper solution, equal to unity.

The rule can only be applied in what is called the "simple case," viz. when the equations can be so written that the lowest terms when equated to zero have no proper solution. For example, the multiplicity of the solution $x_1=x_2=0$ in the equations $x_1-x_2+x_1^2=0$, $x_1^2-x_2^2+2x_1^2x_2=0$ is 4; for, on multiplying the first by $x_1+x_2-x_1^2$, and subtracting from the second, we get $x_1^4=0$, which may be taken in place of the second equation. The equations being written in any definite order, each equation may be modified as above by means of previous equations, but not by means of succeeding equations, since such modification does not affect the number or the multiplicity of the finite solutions. In the case of the equations $x_1^2-x_2^2+x_1^3=0$, $x_1^3-x_2^3+x_2^4=0$, the proviso cannot be fulfilled and the rule cannot be applied. This example belongs to what is called the "general case"; but all the solutions of the system of equations (1) to $(n+2)$ come under the "simple case."

We can now proceed to find all the irrelevant solutions of the equations (1) to $(n+2)$. Consider first the infinite solutions. On putting $u=0$, the equations become $x\Sigma\frac{t}{r}=y\Sigma\frac{t}{r}=0$, $x^2+y^2=r_1^2=\dots=r_n^2$. Hence either $x=y=0$, which gives no proper solution; or else $\Sigma\frac{t}{r}=0$, which requires all the r's to vanish, unless one of the expressions $t_1 \pm t_2 \pm \dots \pm t_n$ vanishes, which we shall assume not to be the case. Thus there are only two different infinite solutions, viz. $u=r_1=r_2=\dots=r_n=0$, $x=\pm\iota y$.

To find the multiplicity of the infinite solution corresponding to $x+\iota y=0$, put $x+\iota y=v$ and $x-\iota y=1$, thus making the system non-homogeneous with $v, u, r_1, \dots r_n$ as unknowns, and

$$u=v=r_1=\dots=r_n=0$$

as the solution whose multiplicity is required. Now from equations (3), (4) we can express u and v as quadratic functions of u, v, r_1, r_2, and by substituting them in equations (5) to $(n+2)$, the lowest terms in these become of degree 2. Also (1), (2) become $(v+1)\Sigma\frac{t}{r}=2u\Sigma\frac{at}{r}$, $(v-1)\Sigma\frac{t}{r}=2\iota u\Sigma\frac{bt}{r}$, of which the second may be replaced by $v\Sigma\frac{t}{r}=u\Sigma\frac{(a+\iota b)t}{r}$, which is of degree $n+1$ when made integral, and when for u, v are substituted the quadratic functions to which they are equal. The product of the degrees of the lowest terms in the equations is now

$$(n-1)(n+1)2^{n-2}.$$

This is the multiplicity of each of the infinite solutions. The number of finite solutions is therefore

$$n^2 2^n-(n^2-1)2^{n-1}=(n^2+1)2^{n-1}.$$

As regards these we may put $u=1$, since $u \neq 0$.

The irrelevant finite solutions are of two kinds. There are 2^{n-1} solutions for which $x=a_p$, $y=b_p$, $r_p=0$; for these values satisfy equations (1), (2), $(p+2)$, and the other equations have 2^{n-1} solutions. Each of these solutions is of multiplicity 2, owing to equation $(p+2)$. This accounts for $n2^n$ irrelevant solutions. Again there are 2^{n-1} finite solutions for which $r_p=r_q=0$; for these values satisfy (1), (2), and leave all the other equations quadratic except one of the equations $(p+2)$, $(q+2)$; which give a linear equation by subtraction.* This accounts for $\frac{1}{2}n(n-1)2^{n-1}$ irrelevant solutions. Assuming that this exhausts the number of irrelevant solutions we find the number of relevant solutions to be

$$(n^2+1)2^{n-1}-n2^n-n(n-1)2^{n-2}=(n-1)(n-2)2^{n-2}.$$

The number of solutions for x, y is only half of this, since to each solution $x, y, r_1, r_2, \ldots r_n$ there corresponds another x, y, $-r_1, -r_2, \ldots -r_n$, in which x, y, have the same values. The final result is that *the total number of solutions of the problem for n co-planar forces is* ${}_{n-1}C_2 2^{n-2}$. I have verified by other methods that this number is correct when $n=3$, 4, or 5, which goes to show that the method of exhaustion followed above is complete. The number is the total number of solutions, real and imaginary, for all the 2^{n-1} combinations of thrusts and tensions among the given forces. The average number of solutions for each combination is therefore $\frac{1}{4}(n-1)(n-2)$.

When the given points are co-planar there is a case of exception in which the number of solutions is infinite. This occurs when the points all lie on a circle, and a point on the circle, other than one of the given points, is a position of equilibrium. In that case every point on the circle is a position of equilibrium. Also (when the points are on a circle) if there is any real position P of equilibrium outside the plane, then every point on a certain circle through P is a position of equilibrium. This circle and the given circle lie in perpendicular planes, and each cuts the plane of the other in a pair of inverse points.

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*This diminution to half the normal number of solutions is accounted for by the fact that the other half are at infinity. The infinite solutions have been previously considered separately.

A similar method can be applied to find the total number of solutions of the problem for n given points in ordinary space. The process is longer and more difficult, and we shall give only a short sketch of it.

In this case we have $n+3$ equations for the ratios of the $n+4$ unknowns $u, x, y, z, r_1, r_2, \dots r_n$, viz.

$$\frac{t_1(x-a_1u)}{r_1}+\dots+\frac{t_n(x-a_nu)}{r_n}=0, \qquad\dots\dots(1)$$

$$\frac{t_1(y-b_1u)}{r_1}+\dots+\frac{t_n(y-b_nu)}{r_n}=0, \qquad\dots\dots(2)$$

$$\frac{t_1(z-c_1u)}{r_1}+\dots+\frac{t_n(z-c_nu)}{r_n}=0, \qquad\dots\dots(3)$$

$$(x-a_1u)^2+(y-b_1u)^2+(z-c_1u)^2=r_1^2, \qquad\dots\dots(4)$$

$$\dots\dots\dots\dots\dots\dots$$

$$(x-a_nu)^2+(y-b_nu)^2+(z-c_nu)^2=r_n^2. \qquad\dots\dots(n+3)$$

These equations have a singly infinite system of solutions, together with a finite number of other solutions which include all the relevant solutions. In geometrical language equations (2) to $(n+3)$ have as their complete intersection a composite curve of degree $n^2 2^n$, part of which lies in (1), and the remaining part, which we shall call the *remainder curve*, cuts (1) in a finite number of points, some giving relevant and some irrelevant solutions. We must first find the degree of the remainder curve.

Part of the intersections of (2) to $(n+3)$ consists of an *infinity curve* of degree 2, which lies in (1), viz the curve

$$u=r_1=r_2=\dots=r_n=0, \quad x^2+y^2+z^2=0.$$

For, on putting $u=0$, we have

$$y\Sigma\frac{t}{r}=z\Sigma\frac{t}{r}=0, \quad x^2+y^2+z^2=r_1^2=r_2^2=\dots=r_n^2.$$

Hence either $y=z=0$, giving 2^n points, none of which lie in (1); or $\Sigma\frac{t}{r}=0$, which requires all the r's to vanish, assuming that no expression of the form $t_1 \pm t_2 \pm \dots \pm t_n$ vanishes. This gives the infinity curve. To find its multiplicity in the intersection of (2) to $(n+3)$ we put $z=1$, and change x, y to $x+x_1$,

$y+y_1$, where $x_1^2+y_1^2+1=0$. From equations (4), (5) we can then express u and xx_1+yy_1 as quadratic functions of u, x, y, r_1, r_2; and by substituting them in (6) to $(n+3)$ the lowest terms in these last become of degree 2. Also equations (2) (3) become $(y+y_1)\Sigma\frac{t}{r}=u\Sigma\frac{bt}{r}$, $\Sigma\frac{t}{r}=u\Sigma\frac{ct}{r}$, the former of which may be changed to $u\Sigma\frac{bt}{r}=(y+y_1)u\Sigma\frac{ct}{r}$, whose lowest terms are of degree $n+1$, when we substitute for u the quadratic function to which it is equal. The infinity curve is therefore of multiplicity $(n+1)(n-1)2^{n-2}$, and counts as of degree $(n^2-1)2^{n-1}$, in the intersection of (2) to $(n+3)$. The complete *finite* intersection of (2) to $(n+3)$ is therefore a curve of degree

$$n^2 2^n-(n^2-1)2^{n-1}=(n^2+1)2^{n-1}.$$

This finite curve includes $\frac{1}{2}n(n-1)$ curves which lie in (1) viz. those for which any two of the r's vanish. On putting $u=1$ and $r_1=r_2=0$, equations (4), (5) are equivalent to one quadratic and one linear equation, while (6) to $(n+3)$ are quadratic. Hence each of the $\frac{1}{2}n(n-1)$ curves is of degree 2^{n-1}, and the *remainder curve* is of degree

$$(n^2+1)2^{n-1}-n(n-1)2^{n-2}=(n^2+n+2)2^{n-2},$$

which cuts (1) in $n(n^2+n+2)2^{n-2}$ points altogether.

All the points in which the remainder curve cuts (1) on the infinity curve and the $\frac{1}{2}n(n-1)$ curves give irrelevant solutions, and also a few other points. The remainder curve meets infinity $(u=0)$ in $(n^2+n+2)2^{n-2}$ points, of which only 2^n (corresponding to $y=z=0$) are not on the infinity curve. The other points lie on the infinity curve; but at each of these points the remainder curve touches $u=0$, so that it has only $(n^2+n-2)2^{n-3}$ points of intersection with the infinity curve. At each of the $(n^2+n-2)2^{n-3}$ points the remainder curve meets (1) in a point of multiplicity $n+1$. This accounts for $(n+1)(n^2+n-2)2^{n-3}$ irrelevant infinite solutions.

At any finite point at which three of the r's vanish, the complete intersection of (2) to $(n+3)$ has four branches, three of which belong to curves for which two of the r's vanish, and the fourth to the remainder curve. There are 2^{n-2} such finite points at which any one set of three r's vanish, and at each the remainder curve meets (1) in a point of multiplicity 2. This accounts for $\frac{1}{3}n(n-1)(n-2)2^{n-1}$ irrelevant finite solutions.

The remainder curve also cuts the curve corresponding to $r_p=r_q=0$ in 2^{n-1} finite points at which no other r vanishes, viz. the points for which $\frac{y-b_p}{z-c_p}=\frac{y-b_q}{z-c_q}=\frac{b_p-b_q}{c_p-c_q}$. This accounts for $\frac{1}{2}n(n-1)2^{n-1}$ irrelevant solutions.

Also the remainder curve meets (1) in 2^{n-1} points corresponding to $x=a_p$, $y=b_p$, $z=c_p$, $r_p=0$, each of which is of multiplicity 2. This accounts for $n2^n$ irrelevant solutions.

Hence we find the number of relevant solutions to be

$$\begin{aligned}&n(n^2+n+2)2^{n-2}-(n+1)(n^2+n-2)2^{n-3}\\ &\qquad\qquad -\tfrac{1}{6}n(n-1)(n-2)2^{n-1}-n(n-1)2^{n-2}-n2^n\\ &=\tfrac{1}{6}(n+3)(n-1)(n-2)2^{n-2}\\ &=\tfrac{1}{6}(n-1)(n-2)(n-3)2^{n-2}+(n-1)(n-2)2^{n-2}.\end{aligned}$$

The number of solutions for x, y, z is half of this.

Hence the total number of solutions of the problem for n forces in ordinary space is ${}_{n-1}C_2 2^{n-2}+{}_{n-1}C_3 2^{n-3}$.

This result is correct also if the given points are all in one plane, for then the total number of solutions in the plane is ${}_{n-1}C_2 2^{n-2}$, and out of the plane ${}_{n-1}C_3 2^{n-3}$. This enables us to guess the answer to the problem for n forces in space of m dimensions. We may in fact assume the answer to be

$${}_{n-1}C_2 2^{n-2}+{}_{n-1}C_3 2^{n-3}+\ldots+{}_{n-1}C_m 2^{n-m}.$$

For $n+1$ forces in space of n dimensions the number of solutions is $2(3^n-\overline{n+2}\,2^{n-1})$, a result which can be verified in other ways.

As regards the number of real solutions, when the given points are not collinear, I can only mention two results: (i) that for four points in ordinary space the fourteen solutions can all be real; (ii) that for $n+1$ points in space of n dimensions, when the distance between every pair of points is the same, and the forces are all equal, there is one and only one real solution for each combination of thrusts and tensions. If, however, $n+1$ is even, the solutions corresponding to an equal number of thrusts and tensions are at infinity and of multiplicity 3.

In the case of a tetrahedron of which the base is equilateral, and the three faces have the same inclination 2α to the base, if the force to (or from) the vertex is t_0, and to (or from) each of the base angles is t, there will be fourteen real solutions provided t_0/t lies between 1 and $3\sin\alpha(1-3\sin^2\alpha\cos^2\alpha)^{\frac{1}{2}}$. (Thus for a rectangular tetrahedron t_0/t must lie between $\cdot 975\ldots$ and 1). When all four forces are tensions, or when three are tensions and one a thrust, there is one solution in each case, and when two are tensions and two are thrusts there are three solutions in each case.

When the given points are collinear the number of real positions of equilibrium in any plane through the line of collinearity can be stated completely, provided no expression of the form $t_1 \pm t_2 \pm \ldots \pm t_n$ vanishes. Let $A_1, A_2, \ldots A_n$ be the given points in order, and $t_1, t_2, \ldots t_n$ the forces (given in magnitude and sign) directed towards them. As the sign of one force may be chosen either positive or negative, we *assume that t_1 is positive.* Then the number of real positions of equilibrium on each side of the line is the excess of the number of positive maxima over the number of positive minima in the series of quantities

$$(-S_n),\ S_1,\ S_2, \dots S_n,\ (-S_1),$$

where

$$\begin{aligned} S_1 &= -t_1 + t_2 + \dots + t_n = \Sigma t - 2t_1, \\ S_2 &= -t_1 - t_2 + \dots + t_n = S_1 - 2t_2, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ S_n &= -t_1 - t_2 - \dots - t_n = S_{n-1} - 2t_n. \end{aligned}$$

By a maximum (or minimum) quantity is meant one which is greater (or less) than each of the adjacent quantities. The rule is applied to $S_1, S_2, \dots S_n$ only, S_1 and S_n being counted as minima if less than the adjacent quantity, but not as maxima if greater than the adjacent quantity, since $S_1 < -S_n$, $S_n < -S_1$.

Thus if there are $2n+1$ equal forces alternately pushing and pulling, there are n real positions of equilibrium on each side of the line.

If $t_1, t_2, \dots t_n$ are the absolute magnitude of the forces, and positive, the total number of real positions of equilibrium for all possible combinations of thrust and tension is the excess of the total number of positive signs over the total number of negative signs in all the expressions $-t_1 \pm t_2 \pm \dots \pm t_n$ which have positive values. This result (like the last) is independent of the distances between the given points, and (unlike the last) remains unchanged if the order of the points is changed.

The greatest possible number of real positions of equilibrium is found by subtracting each coefficient in the expansion of $(1+x)^{n-1}$ from the greatest coefficient, and adding all the results. This is counting the solutions on both sides of the line; and may be compared with the total number of solutions, real and imaginary, viz. $(n-1)(n-2)2^{n-3}$. The ratio is roughly that of 3 to $n^{\frac{3}{2}}$.

F. S. MACAULAY.

REVIEWS.

The Theory of Sets of Points. By W. H. YOUNG, M.A., Sc.D., and GRACE CHISHOLM YOUNG, Phil. Doc. (Gött.). (Cambridge: at the University Press. 1906. Pp. xii + 316.)

The theory of aggregates may roughly be divided into two parts: the one, which arose out of the profound study of integration, is associated with the names of Riemann, Hankel, Smith, Harnack, P. du Bois-Reymond, Lebesgue, and Young; while the other, which arose from the need for a secure basis of the theory of analytic functions, has, as characteristics, Weierstrass' and Cantor's theories of irrationals, Cantor's theory of derivatives of point aggregates, Cantor's theory of the 'powers' (*Machtigkeiten*) of aggregates, Cantor's theory of transfinite numbers, and the modern developments of this last theory. The two parts are by no means disconnected; thus, Cantor has greatly contributed to the theory of the "content" of an aggregate (the essential notion in the conception of integration), and Dr. Young, whose past and present work has been mainly concerned with what may be called the Riemann-direction, has given some excellent chapters on Cantor's ideas and numbers.

To quote the author's preface: "In subjects as wide apart as Projective Geometry, Theory of Functions of a Complex Variable, the Expansions of Astronomy, Calculus of Variations, Differential Equations, mistakes have in fact been made by mathematicians of standing, which even a slender grasp of the Theory of Sets of Points would have enabled them to avoid." This should raise interest in the theory even among mathematicians who do not concern themselves with fundamental questions, while, for those who do, this theory has, of course, shown itself to be indispensable.

On account of both the novelty of the subject as material for a text-book and the independence and originality of its treatment, the work of Dr. and Mrs. Young deserves the highest praise; in this notice, I have thought it would be more useful, instead of praising the many excellences of the book in detail, to direct attention to those few parts which, in my opinion, require modification or discussion.[*]

The correspondence between the real numbers and the points of a straight line is dealt with on pp. 9-15. By a fairly well-known extension of von Staudt's method of forming a projective number-scale, combined with the Cantor-Dedekind axiom, the result is reached (p. 14) that the continuum of real numbers and the linear continuum can be brought into a (1, 1)—correspondence, maintaining the order. Two points should be remarked. Firstly, Cantor stated his axiom (the date on p. 14 should be 1872 instead of 1878) in a form applying to the *metrical* geometry of the straight line. Secondly, all we know of that projective scale is that it is of type θ,[†]—and series of *some* (which are not *all*) real numbers can be of this type (Harnack, Bettazzi[‡]). Thus it seems that purely ordinal characteristics do not describe our notion of the straight line, but that a line (as a substratum to a point-set) is a set of points in *metrical* (not merely ordinal) correspondence with the system of real numbers. The difference between sets of points on a substratum and a set of intrinsically determined order is emphasised on pages 22-23, 127, and 287.

In the definition (p. 16)[§] of what is meant by a 'set of points,' we read: "a linear set of points consists of points of a straight line determined by a certain law which is such that (1) every point of the straight line either belongs to the set or does not, but not both, nor neither; . . . ." This condition seems so self-evident to the student that its meaning cannot be grasped at this stage. The importance of the condition lies in the fact that we can frame certain 'definitions' which *appear* to define valid classes ('sets'), and yet such a 'set' both is and is not a member of itself. Such 'sets' are Russell's 'class of the x's such that x is not a member of itself' and the class known as the 'cardinal number of all the ordinal numbers.'[*]

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[*] There is (pp. 1-5) a short account of the system of real numbers as defined by (Cantor's) 'sequences.' I hope to give, in the near future, a brief discussion of what one can mean by the phrase 'definition of a number by a sequence' in this *Gazette* (cf. Russell, *The Principles of Mathematics*, vol. i., 1903, pp. 270-286).

[†] Cf. Russell, *op. cit.*, pp. 353, 384 *sqq.*

[‡] *Ann. di mat.*, t. XVI. (1888-9), pp. 49-60.

[§] An analogous definition is on p. 170.

It is logically better to separate the idea of 'potency' (power, or cardinal number) from that of measurement (cf. pp. 33, 76), for, as Russell[†] has shown, the latter idea is irrelevant to this part of mathematics, at least.

In the description (pp. 124-127) of Bernstein's diagrammatic representation of simply-ordered enumerable series, reference should, perhaps, be made to the prototype of this,—Schröder's representation by a 'matrix' of a binary relative.[‡]

There seems to be some confusion in the note on p. 159. The number ω^{ω} is not the number ω exponentiated by ω in any sense analogous to exponentiation as defined for cardinal numbers. The definitions of ordinal exponentiation given by Hausdorff and myself,[§] on the contrary, are.

In Ex. 5 on p. 152, the symbolism $\underset{n=\infty}{\mathrm{Lt}}\, n!$ is wrong; for the meaning of this is: The least cardinal which is greater than all the (finite) cardinals $n!$; and this is a; whereas what we want is the product of *all* the finite cardinals, and this is equal to c.

In the discussions of the law of arbitrary choice and the multiplicative axiom (pp. 147, 289-291), it appears to me that Bernstein's conception of 'multiple equivalence' is irrelevant. Given any not-null class u, we can state certain propositions beginning 'x is a u,' and it is irrelevant in how many ways x can be chosen: the multiplicative axiom is necessary in order to be able to say that '$x, y, z, \ldots$ (an infinity) are different members of u' has any meaning at all (given, of course, that u is infinite). A case of this axiom, it should be stated, is involved in the definition (p. 151) of the product of an infinity of cardinals.

There are some small misprints: p. 19 (note), for 1881 read 1882, and for 1892 read 1882; pp. 124, 134, 301, 'Huntington' is spelt wrongly; on p. 155, for 2ω read $\omega . 2$, and so on; on page 191 (note) the reference to Bolzano should be "Rein analytischer Beweis...," 1817 (lately republished, with notes, in Ostwald's *Klassiker der exakten Wissenschaften*); on p. 295, 'Lebesgue' is spelt wrongly, and on p. 299, 'Beiträge'; and (p. 155) a number is not a 'symbol.'

In the admirable Bibliography (pp. 295-309), it should, I think, be remarked that the French translations of Cantor in vol. 2 of *Acta Math.* are not always full; that the work mentioned in the 10th reference to Cantor is included in that mentioned in the 9th; and references to Dedekind's "Stetigkeit und irrationale Zahlen" of 1872, G. H. Hardy's important paper on absolutely convergent series in *Proc. Lond. Math. Soc.*, for 1905, Pringsheim's article on irrational numbers in the *Encykl. der Math. Wiss.*, and Russell's papers in *Mind of* 1905 (on "Denoting") and *Proc. Lond. Math. Soc.*, for 1906, should be given.

PHILIP E. B. JOURDAIN.

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[*] Cf. pp. 145-147, especially the suggestive ll. 16-25 on p. 146.

[†] *Op. cit.*, pp. 111-116.

[‡] *Algebra und Logik der Relative*, Leipzig, 1895, pp. 42-68.

[§] *Mess. of Math.*, May, 1906, pp. 13-16.

Tableaux logarithmiques. A et B. By Dr. A. Guillemin. Pp. 32 of explanatory matter, with two pockets containing the two tables. Paris, Félix Alcan, 1906. Price, 4 francs complete. 2 francs for separate parts.

Let N be a number whose logarithm is required, and let m be a near number whose logarithm is given in a table. Then if $N=m(1+a)$ the quantity to be added to the known $\log m$ in order to obtain the required $\log N$ is $\log(1+a)$, and is a quantity of the same order of magnitude as a, when a is small.

This property is utilised by Dr. Guillemin to enable logarithms and antilogarithms to be obtained to 6 and 9 decimals respectively by means of his tables A and B, each table containing only 1,000 entries, and the larger table measuring only roughly 1 ft. 6 in. × 1 ft. 3 in. The tables are really tables of antilogarithms rather than of logarithms. The logarithms in the second column (L) are the successive decimal numbers of three figures from 000 to 999; the corresponding antilogarithms in the first column are given to six places in Table A and to nine places in Table B. The third columns give the values of $\log a$ to 3 and 6 places respectively, for which the corresponding values of $\log(1+a)$ are entered in the second columns.

Suppose then I want to find $\log 3\cdot141593$ from Table A to six places. The nearest number in the table is 3140509, of which the logarithm is given as ·497, and these are the first three figures of the required logarithm. The process stands thus:

$$N=3\cdot141593 \qquad \log N=\log m+\log(1+a)$$
$$m=3\cdot140509 \qquad \log m=\cdot497$$

By subtraction

$$N-m=ma=0\cdot001084.$$

I then look for $\log 0\cdot001084$ in the tables and find $\log 1083927=\cdot035$, which is the value required for $\log ma$ *to three decimal places.* The work now stands thus:

$$\begin{aligned}\log ma &= 3\cdot035\\ \log m &= \ \cdot497 \text{ (above)},\\ \therefore\ \log a &= 4\cdot538.\end{aligned}$$

I then look for $\bar{4}\cdot538$ in the column headed $\log a$ and find it opposite 150 in the column L. This means that the corresponding value of $\log(1+a)$ is ·000150, and adding this to the value of $\log m$ the final result is $\log 3\cdot141{,}593=\cdot497150$.

In the second table (Table B) the antilogarithms are given to 9 decimal places and the values of $\log a$ to 6 decimal places. To obtain logarithms correct to 9 decimal places it might be thought at a first glance that it would only be necessary to repeat twice the operations required with six figure work with Table A. An attempt to do so, however, shows that the method is not so simple as this, but that it is necessary to perform the three operations described by the author in his explanation.

Suppose, for example, it is required to calculate $\log 5$.

The tables give $0{\cdot}698 = \log 4{\cdot}998{,}884{,}875$, so that 0·698 are the first three figures of $\log 5$ with a remainder in the antilogarithm of 0·011,155,125. If we use the $\log a$ method *either* with Table A *or* with Table B there is not much difficulty in obtaining the next three figures 970, but the table does not contain sufficient data to find the final remainder required to give the last three figures. The only way is to use the tables *backwards* in order to obtain the antilogarithm of 0·698,970 correct to 9 decimals. This involves a second use of the tables with the assistance of the $\log a$ column, and leads to the result

$$\log 4{\cdot}999{,}999{,}954 = 0{\cdot}698{,}970.$$

The final remainder is 0·000,000,046, from which a repetition of the first operation gives the last three figures of the logarithm, namely 004.

It will thus be seen that the calculation of logarithms to 9 figures is three times as difficult as their calculation to 6 figures. It involves three distinct operations instead of one, and the author is therefore justified in calling the tables A and B respectively single and triple extension tables. It is not often that logarithms are actually required to 9 places, and the emergency may be provided for by keeping a copy of the tables and learning to work Table B when necessity arises, rather than by keeping a big cumbersome book of tables which is never to be found when it is wanted, and still involves cumbersome work in the matter of interpolation.

It should be pointed out that the class teacher who sends the copies of the explanatory notice to the bookbinder to have the pages cut in order to prevent his pupils from wasting their time, will find the pockets for holding the tables cut off. French people are as a rule mortally afraid of the guillotine, but unless this is used the pages should be made larger than the pockets, which they are not.

G. H. BRYAN.

Einführung in die Theorie der Differentialgleichungen mit einer unabhängigen Variabeln. Von DR. LUDWIG SCHLESINGER. (*Sammlung Schubert*, XIII.) (Leipzig, 1904.)

Gewöhnliche Differentialgleichungen beliebiger Ordnung. Von DR. J. HORN. (*Sammlung Schubert*, L.) (Leipzig, 1905.)

The subject of differential equations has grown to such an extent that it is necessary for the writer of a text-book to confine himself to a particular aspect of the subject. The works that we have before us deal with what might be called the formal side of the subject, that is the considerations connected with the existence of solutions, the nature of the integrals in the vicinity of the singularities, and the form of the equations when the singularities are of a specified type.

Dr. Schlesinger's treatise is a good introduction to the higher work on the subject. The writer, aiming at simplicity, considers only equations of the first two orders, a feature which will be welcome to many readers since it enables them to obtain a grip of the essential ideas without having to master any difficult analysis.

It is a significant fact that the differential equations which arise out of problems in geometry and physics are no other than the equations with the simplest types of singularities; this point is brought out clearly in Schlesinger's treatise, Riccati's equation for instance being obtained when the differential equation

$$\frac{dy}{dx}=f(x, y)$$

is supposed not to possess an integral with a variable branch point.

Another good feature which might be noticed is that in obtaining the definite integral solutions of the various equations the writer calls attention to the fact that in each case the method depends upon the construction of a relation of the form

$$D_z\{u(x, z)\}=\Delta_z\{v(x, z)\},$$

and we believe that it will be through the development of the theory of relations of this type that further progress in this branch of the subject may be expected.

Dr. Horn's aim has been to write a more elaborate treatise on the subject avoiding as much as possible a repetition of the portions treated by Schlesinger. The characteristics of the work are the frequent occurrence of systems of linear differential equations and the use that has been made of the canonical forms of a linear substitution.

In dealing with a system of ordinary linear differential equations with constant coefficients

$$\frac{dy_r}{dx}+\sum_{s=1}^{n} a_{rs} y_s=f_r(x) \quad (r=1, \ldots n)$$

the writer employs Weierstrass' method which depends upon the reduction of the system to the canonical form

$$\begin{aligned} \frac{dY_1}{dx}&=rY_1+F_1(x), \\ \frac{dY_2}{dx}&=rY_2+Y_1+F_2(x), \\ \frac{dY_n}{dx}&=rY_n+Y_{n-1}+F_{n-1}(x). \end{aligned}$$

This process is more scientific than the usual one since it avoids the use of operators and is in keeping with the general plan of the work; we do not think, however, that it is quite so convenient in actual practice as the symbolic method.

A good feature of the book is that it gives an account of much recent work that has been done on the subject; thus, besides the valuable chapters on the asymptotic representation of an integral and infinite determinants, we find that mention is made of Painlevé's canonical forms of differential equations of the second order and of a certain type whose integrals are uniform functions.

The writer also does well to call attention to theorems that are related to some of the recent mathematical work, as for instance the theorem that if the coefficients $P_i(x, \mu)$ $(i=1, \ldots n)$ of the linear differential equation

$$\frac{d^n y}{dx^n}+P_1(x, \mu)\frac{d^{n-1}y}{dx^{n-1}}+ \ldots P_n(x, \mu)y=0$$

are continuous functions of the real variable x in the interval $|x-a| \leqq r$ and whole rational functions of the complex parameter μ. A solution y which is such that the initial values of $y, y' \dots y^{(n-1)}$ for $x=a$ are independent of μ is a whole transcendental function of μ for values of x within the given range (see p. 297), a theorem which corresponds to an important result in the theory of integral equations.

The printing is good in both cases and I have not noticed any misprints.

H. Bateman.

MATHEMATICAL NOTES.

205. [L'. 2. b]. [A solution, not by elliptic functions is wanted of the following:

Given five lines *abcde* in a plane, it is known that the pairs of points *ab*, *ce*; *bc*, *da*; *cd*, *eb*; *de*, *ac*; *ea*, *bd* are in a collineation. Prove that the fixed triangle of this collineation is self polar as to both the conic on (*ab*, *bc*, *cd*, *de*, *ea*) and the conic on (*ac*, *ce*, *eb*, *bd*, *da*). (F. Morley).]

Solution by H. Bateman.

Lemma. Let a collineation referred to its fixed triangle be given by the equations

$$x' = ax, \quad y' = by, \quad z' = cz,$$

then the conic

$$S' \equiv Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

will correspond to

$$S \equiv Aa^2x^2 + Bb^2y^2 + Cc^2z^2 + 2Fbcyz + 2Gcazx + 2Habxy = 0,$$

and a point $P(\xi, \eta, \zeta)$ will correspond to the point $P''\left(\frac{\xi}{a}, \frac{\eta}{b}, \frac{\zeta}{c}\right)$ on the one hand and to $P'(a\xi, b\eta, c\zeta)$ on the other.

The polar of P'' with regard to S is

$$Aax\xi + Bby\eta + Ccz\zeta + F(c\eta z + b\zeta y) + G(c\xi z + a\zeta x) + H(a\eta x + b\xi y) = 0,$$

and the polar of P' with regard to S' is

$$Aax\xi + Bby\eta + Ccz\zeta + F(b\eta z + c\zeta y) + G(a\xi z + c\zeta x) + H(b\eta x + a\xi y) = 0.$$

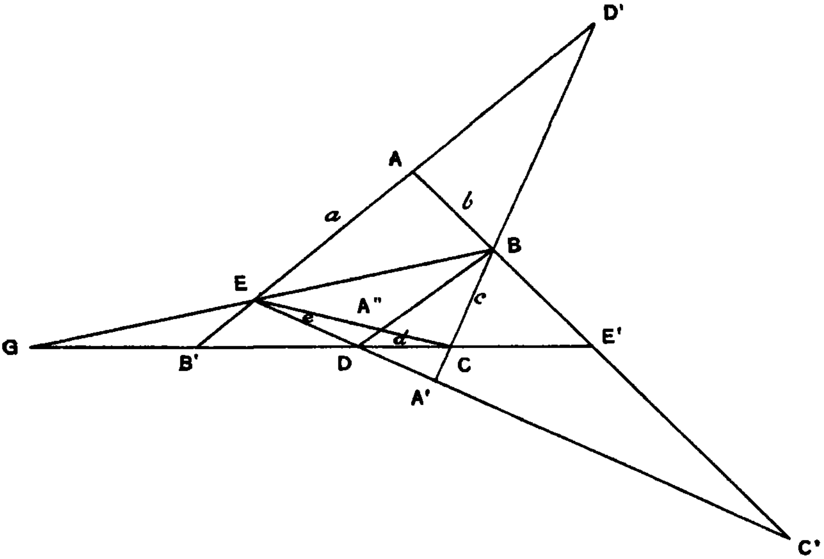
These lines will be the same if

$$\frac{Aa\xi + Ha\eta + Ga\zeta}{Aa\xi + Hb\eta + Gc\zeta} = \frac{Hb\xi + Bb\eta + Fb\zeta}{Ha\xi + Bb\eta + Fc\zeta} = \frac{Gc\xi + Fc\eta + Cc\zeta}{Ga\xi + Fb\eta + Cc\zeta} = \lambda, \text{ (say)}.$$

Eliminating ξ, η, ζ we get a cubic for λ, hence there will in general be three positions of P for which the lines are the same; if there are more than three positions of P, we must have $F = G = H = 0$, and then the conics S and S' will have the fixed triangle as a self-conjugate triangle, and the two lines will be the same for all positions of P.

To apply this lemma to the theorem in question, let $ABCDE$ be the pentagon formed by the five lines *abcde*, $A'B'C'D'E'$ the pentagon formed by the alternate sides. We can show at once that A, A'; B, B'; C, C'; D, D'; E, E' are in a collineation, by comparing the cross-ratios of the pencils $A\{BCDE\}$, $A'\{B'C'D'E'\}$, and the corresponding pencils for the other corners.

Let S be the conic $ABCDE$, S' the conic $A'B'C'D'E'$. Now the two points which correspond to A in the collineation are A'' on the one hand (for EC corresponds to $E'C'$ and BD to $B'D'$) and A' on the other; also the polar of A'' with regard to S is easily seen to be the line $A'G$, and by considering the Pascal hexagon $A'D'B'E'C'A'$ we see that this line is the tangent at A' to the conic S'; accordingly the polar of A'' with regard to S is the same line as the polar of A' with regard to S'. Taking the five points $ABCDE$



in turn, we have five pairs of points for which this condition is satisfied; hence from the above lemma we conclude that the conics S and S' must have the fixed triangle of the collineation as a self-conjugate triangle.

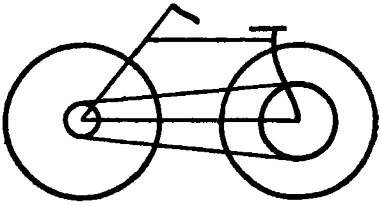
206. [D. 2. a.]. From Prof. Elliott's remark, I gather that he *presupposes* Dedekind's "Schnitt" theory. Of course, if this is done, and thus real numbers are defined (as 'Schnitte'), the sufficiency of the criterion of convergence can be proved, and this was done by Dedekind himself, at the end of his *Stetigkeit und irrationale Fahlen*, more shortly and, I think, more simply than in Prof. Elliott's note.

But I thought that Prof. Elliott was attempting to prove the sufficiency of the criterion *without* any previous arithmetical theory of irrationals (since he did not mention any), and I pointed out the impossibility of this.

PHILIP E. B. JOURDAIN.

207. [I. 4.]. "*Graphs.*"—If in many examinations you set a question on tracing a simple conic, say, $x^2-y^2+x+y-1=0$, you are ruled out because "General Equation of the Second Degree is not in the syllabus." The remedy is to replace the simple conic by a difficult *cubic* curve, the question then becomes a "graph" and is accepted without opposition. G. H. BRYAN.

208. [I. 4.]. *Figure of a Bicycle.*—The annexed figure or something like it was drawn by a candidate in an examination in Applied Mathematics in a question on Rigid Dynamics. The writer *believes* that that candidate passed. G. H. BRYAN.



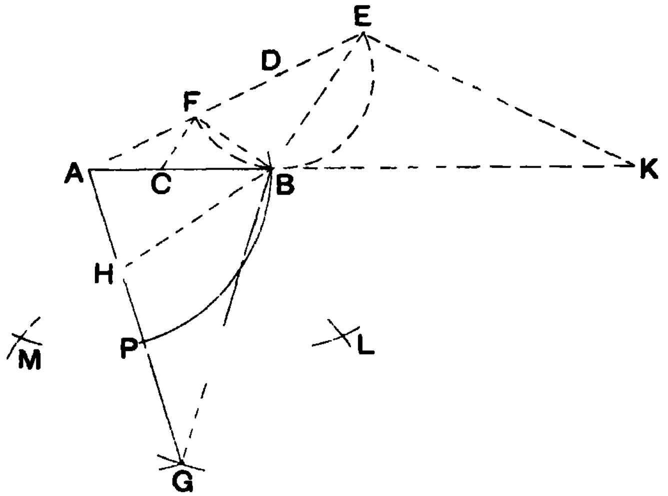
209. [V. a.] Mr. Trachtenberg (195 [X. 1]) suggests the use of $\frac{292}{89}$ feet as the value of a metre correct to 5 places of decimals. Unfortunately, though it seems mere scholastic pedantry to mention it, this value is founded on the old legal value of a metre as 39·37079 inches and not on the new value 39·37011 inches. Many text-books and examination papers fall into the same error.

F. W. Dobbs.

210. [K. 9.] I. *Construction of a regular pentagon.*

Mr. A. F. Van der Heyden asks for a simple line of argument with regard to the construction of a pentagon. Wishing to keep the construction, and being unwilling to give much space to it, I worked out the following method which is given in substance in pp. 71, 77, 78 of my "Geometry." It is as simple as any in construction, simpler than most in argument; it is entirely geometrical, and moreover affords excellent practice in similar triangles.

If AB is divided at C so that CB is the mean proportional of AB, AC, then I have called CB the *mean part* of AB (this is clearly a better name than *medial*) and the triangle AGB of Eucl. IV. 10 I have called the *mean base triangle.* AB is also said to be divided *in mean* at C.



If AB is any line, perp. $BD=\frac{1}{2}AB$, $AFDE$ drawn to meet the circle centre D, radius DB; then $FE=2DB=AB$.

Also angle $ABF=AEB$ in opposite arc, and triangle $AFB \;|||$* triangle ABE.

$$\therefore \frac{AF}{FE}=\frac{AF}{AB}=\frac{AB}{AE}=\frac{FE}{AE}.$$

$\therefore$ FE or AB is the mean part of AE.

Draw $FC \parallel BE$; $\therefore$ CB is the mean part of AB (similar division).

We have constructed both CB the mean part of AB, *and* AE *of which* AB *is the mean part.*

Make $AG=BG=AE$; then ABG is the mean base triangle, which I have shown has angle $A=B=2G$.

It is easier however to suppose $A=B=72°=2G$.

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* ||| "is similar to."

Draw BH bisecting angle B; $\therefore$ triangle HBG is isosceles, and triangle $ABH \,|||\, BAG$; (angles are 72°, 36°, 72°)

$$\therefore\ HG = HB = AB.$$

Also since BH bisects angle B in triangle ABG

$$\frac{AH}{HG} = \frac{AB}{BG} = \frac{HG}{AG};$$

$\therefore$ HG, *i.e.* AB, is the mean part of AG, and $\therefore$ $AG = AE$.

To construct pentagon: If AB is radius of circle, construct triangle ABG (this is very quickly done), and arc BP, centre A; then BP is the side of the in-pentagon.

If the side AB is given, construct triangle ABG; and construct vertices L, M by arcs centres B, G, A, radius AB.

Then figure $BHGL$ is a rhombus;

$$\therefore\ \text{angle } BLG = BHG = 108° = \text{(similarly) } AMG,$$

$$\text{angle } ABL = 3 \times 36° = 108° = \text{(similarly) } BAM \text{ and } MGL;$$

$$\therefore\ ABLGM \text{ is a regular pentagon.}$$

The construction is quickly written out. 'Construct AE of which AB is the mean part, construct isosceles triangle ABG, having $AG = BG = AE$, etc.' Or 'Construct the mean base triangle ABG, base AB, etc.'

The line EK parallel to FB determines the external point K such that BK is the mean proportional of AB, AK. This is distinctly better than the circle construction from centre A, radius AE, since there is no chance of getting the wrong point, and no proof is required.

Another feature entirely wanting in the traditional construction is that by continuing the series of parallels to EB, FB from C and K, we determine an endless series of mean parts (since the successive triangles vertex A are similar).

Thus $\qquad AC = \text{mean part of } AF$

$\qquad\qquad = \text{mean part of mean part of } AB$

$\qquad\qquad = (\text{mean part})^3 \text{ of } AE$

$\qquad\qquad = (\text{mean part})^4 \text{ of } AK$, etc.

It is at once seen also that if CB is the mean part of AB, then AC is the mean part of CB; or if b is the mean part of a, then $a - b$ is the mean part of b, and $2b - a$ the mean part of $a - b$, $2a - 3b$ of $2b - a$ and so on continually.

For AF is the mean part of AB and $= CB$,
and AC „ „ AF *i.e.*, of CB. E. BUDDEN.

211. [V. a.] *Shortened Method of Multiplication, etc.*

I most heartily endorse Mr. Godfrey's objection to these, particularly in multiplication. The shortened method of multiplication is not only longer, but from the nature of the case more inaccurate, as it cannot be tested by the rule of nines; and mere reworking may easily repeat a mistake once made. The stopping in division when enough figures are reached is obvious and is sufficient. A slight fraction of time is saved in taking square roots to 5 figures (which are often wanted to get 3 correct in quadratics), by getting the last two by division. This is, I am sure, all that is worth while teaching; moreover it can easily be taught as the original rule, and there is no difficulty with the decimal point. Whenever I have to teach the shortened method in multiplication I always think of the new shorthand clerk who was taking down notes in longhand, and when asked if he could not write shorthand, said 'Yes, but it takes me longer to write than the other.' E. BUDDEN.

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Wanted.

Mathematical Gazette. 1st Quarto Number. T. J. Garstang, M.A., Bedales School, Petersfield, Hants.

BOOKS, ETC., RECEIVED.

A School Geometry. Parts I. and II. *With Introductory Course of Experimental and Practical Work.* By H. S. Hall and F. H. Stevens. Pp. viii, 138, iv. 2s. 6d. 1906. (Macmillan.)

The Theory of Sets of Points. By W. H. Young and Mrs. Grace Chisholm Young. Pp. xii, 316. 12s. net. 1906. (Cambridge University Press.)

A Second Geometry Book. By J. G. Hamilton and F. Kettle. Pp. viii, 292. 3s. 6d. 1906. (Arnold.)

Mathematical Drawing. By Prof. G. M. Minchin and Prof. J. B. Dale. Pp. viii, 145. 7s. 6d. net. 1906. (Arnold.)

Text-book of Mechanics. By Prof. L. A. Martin, Jr. Vol I. Statics. Pp. xii, 142. 1906. (Wiley, Chapman and Hall.)

Über die Entwicklung der Elementar-Geometrie im XIX. Jahrhundert. By M. Simon. Pp. viii, 278. 8 m. 1906. (Teubner.)

Zur Ausgleichung der Widersprüche in den Winkelbedingungsgleichungen Trigonometrischer Netze. By L. Krüger. Pp. 34. 2 m. 80. 1906. (Teubner.)

Die Absolute Wahrheit der Euclidischen Geometrie. By A. Lippmann. Pp. 68. 1906. (Gerstäcker, Leipzig.)

Leçons sur les Séries Trigonométriques. By H. Lebesgue. Pp. 128. 3 fr. 50. 1906. (Gauthier-Villars.)

A System of Applied Optics, being a complete system of the second order, and the foundation of a complete system of the third order. With examples of their applications. By H. Dennis Taylor. Pp. xvi, 334. 30s. net. 1906. (Macmillan.)

Tôkyô Sûgaku-Buturigakkwai Kizi-Gaiyô. Vol. III. Nos. 3-5. 1906. (Tôkyô Mathematico-Physical Society.)

On Functions having an Addition Theorem. T. Hayashi. *Harmonic Analysis of the Variation of Latitude.* 1890-1905. H. Kimura. *Strains produced by Surface Loading over a Circular Area.* H. Nagaoka.

Lessons in Experimental and Practical Geometry. By H. S. Hall and F. H. Stevens. Pp. viii, 138, iv. 2s. 6d. 1906. (Macmillan.)

Junior Arithmetic Examples. By W. G. Borchardt. Pp. viii, 171, xl. 1906. 1s. 6d. (Rivingtons.)

Tableaux Logarithmiques A et B, équivalant à des Tables de Logarithmes à 6 *et à* 9 *Décimales.* By A. Guillemin. Pp. 32. 4 frs. 1906. (Alcan.)

Principes et Formules de Trigonométrie Rectiligne et Sphérique. By J. Pionchon. Pp. 146. 5 frcs. 1906. (Gauthier-Villars.)

A Brief History of Japanese Mathematics. By T. Hayashi. Reprinted from *Nieuw Archief voor Wiskunde.*

Elementary Algebra. Part I. By P. Ross. 3s. with answers, 2s. 6d. without. Pp. viii, 272, 43. 1906. (Longmans.)

Examples and Homework in Preliminary Practical Mathematics. By T. I. Cowlishaw. Pp. 53. 1s. 1906. (Longmans.)

Projective Differential Geometry of Curves and Ruled Surfaces. By Prof. E. J. Wilczynski. Pp. viii, 298. 10 m. 1906. (Teubner.)

La Mécanique des Phénomènes fondée sur les Analogues. By Prof. M. Petrovitch. Pp. 95. 2 frcs. 1906. (Gauthier-Villars.) Series *Scientia.*

A Junior Arithmetic (without Answers). By C. Pendlebury and F. E. Robinson. Pp. xii, 204. 1s. 6d. 1906. (Bell.)

Junior Arithmetic (with Answers). By W. G. BORCHARDT. Pp. viii, 221, xl. 2s. 1906. (Rivingtons.)

Wissenschaft und Hypothese. By H. POINCARÉ; translated into German by F. and L. LINDEMANN. Second Edition, Pp. xvi, 346. 4 m. 80. 1906. (Teubner.)

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On two Differential Equations in Lagrange's "Mécanique analytique." By P. E. B. JOURDAIN. Reprinted from *Bibliotheca Mathematica*, May 4th 1906.

Algebra for Elementary Schools. In three stages. By W. M. BAKER and A. A. BOURNE. Pp. 48, 75, 66. 6d. each. 1906. (Bell.)

Algebraic Geometry. By W. M. Baker. Pp. 325, xxiii. 6s. 1906. (Bell.)

Longmans' Complete Arithmetics. Scheme B. Bk. VIII. Pp. 103, 16. 9d. (with answers 1s). 1906. (Longmans.)

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THE

MATHEMATICAL GAZETTE.

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THE PRINCIPLES OF DYNAMICS.

I. The object of this Paper.

(1) The writer is not attempting to construct a new system of dynamics, but merely to present in a clear form what he believes to be the principles of our existing system; and he ventures to hope that such a presentation may be of some use to those teachers of dynamics who are not specialists, and perhaps to older students. The paper is not intended for experts.

It has appeared to him that in the various text-books written on this subject, these principles are at times forgotten; with the result that there are to be found many faults, some due to incompleteness, some to vagueness, some to reasoning in a "vicious circle," and some to the attempt to make one equation give us two or even three unknowns.

(2) As an example of the last, it may be mentioned that in one text-book the second of Newton's laws appears to be regarded (i) as an experimental law giving the observed actions of forces (measured statically) on masses (measured independently of the observed effect); (ii) as defining force; (iii) as defining mass; (iv) as giving us a means of measuring time.

(3) So again there appears to be much vagueness as to the choice of origin and axes, and consequently much uncertainty as to what is meant by velocity and acceleration, by displacement, work, and kinetic energy.

II. The conditions to be fulfilled by a rational system of dynamics.

(1) *Force.* It is easy by means of spring-balances or by some other statical method to determine multiples of any arbitrarily-chosen unit of force. It may be objected that springs change in

properties as time goes on. But what more exact methods are there than statical methods of actually *obtaining* a series of forces of known relations to one another? One may *define* forces dynamically, but one does not *measure* them dynamically in practice.

Even those who profess to define and measure forces dynamically by means of Newton's laws do yet in fact refer back to the statical measurement. If a body does not "persist in its motion with uniform velocity," they are not contented to say "then there must, by definition, be a force acting on it"; but they look about for some other body exerting on the first a force measurable statically. In all statical mechanics we measure forces by the spring-balance or by some equivalent statical method; and in the dynamics of machines we measure statically the forces producing the motions considered. We should not be satisfied with, nor could we work with, any system of dynamics in which the measures of the forces estimated dynamically did not agree with their measures estimated statically.

For this reason, in discussing Newton's laws, the writer will assume that the relative magnitudes of the forces are determined statically. These laws once established, we may then avail ourselves of them to obtain a suitable *definition* of the unit of force.

[In this connection see Thomson and Tait's *Natural Philosophy*, Vol. I., Part I., §§ 207, 248, 258, third paragraph, 220 and 223 [ed. 1879]. Also see XII., (2) in this paper.]

(2) *Mass.* The root-idea of *mass* in dynamics is no doubt *inertia*; *i.e.* the passive resistance offered by matter to change of motion.

Yet—[See Thomson and Tait, Vol. I., Part I., §§ 208, 209, 216]—the Newtonian conception of *mass* was such that its measurement was regarded as independent of dynamical experiment. And there is no doubt that we should not be satisfied with, nor could we work with, any system of dynamics in which dynamical measurement did not give (*e.g.*) to 5 pints of water five times the mass that it gave to 1 pint.

The writer believes, then, that in discussing Newton's laws he is justified in assuming that the masses spoken of are to be thought of as measured, and their centres of mass as determined, independently of any dynamical experiments; though, these laws once established, masses may be defined dynamically.

(3) *Definiteness of meaning and significance.* A further condition to be fulfilled will be discussed in VIII. and IX. Suffice it here to say that we expect our system of dynamics to give us definite and significant results where the data are, as far as we can judge, definite and sufficient.

III. Velocities and angular velocities.

"All motion that we are or can be acquainted with is *Relative* merely." [Thomson and Tait, Vol. I., Part I., § 45.]

"When we turn our attention to the process of change" . . . (in a system) . . . "itself, as taking place during a certain time and in a continuous manner, the change of configuration is ascribed to the motion of the system." [Clerk Maxwell, *Matter and Motion.* Art. xxiv., ed. 1876.]

(1) That it is not "slaying the slain" to point out how meaningless to us is the expression "*absolute velocity*" is shown by the recent appearance in a well-known scientific journal of "A plea for Absolute Motion." ["Nature," Vol. 73, p. 463.]

(2) No one can deny that our minds can set no limit to space; nor can we say "beyond this or that limit there is no matter." Our whole universe of visible stars may be only one of a countless multitude of such universes; the solar system may appear to "move" this way or that through our universe of stars; that universe of stars to "move" this way or that through a cloud of other universes; and so on to infinity. What meaning *can* "absolute motion" convey to our minds?

(3) *Rotation kinematically considered.* So again we can imagine system outside system, each with rotation relatively to its neighbours. But what kinematical meaning *can* we assign to the expression "absolute rotation"?

(4) *Rotation dynamically considered.* At first sight it would appear as though the phenomena of dynamics do give us a conception of absolute rotation.

We believe it to be established that when a mass of homogeneous fluid has rotation relatively to axes which are "fixed" with respect to distant stars, then it assumes, or oscillates about, a spheroidal form; and that when two bodies of matter rotate round each other relatively to the distant stars, in either direction, then the apparent attraction between them, as measured by the strain of a massless spring connecting them, would be less than it would be were there no such rotation.

But what more can we say than this? May not "rotation relatively to distant stars" mean "rotation *relatively* to the ether of our universe"? We cannot answer this.

One result remains. Dynamically we cannot define even arbitrarily what we shall call "absolute velocity"; but dynamically we *can* define what we shall call "absolute rotation," though we cannot deny but that the definition may be quite arbitrary.

(5) "*Fixed*" *axes.* In dynamics we shall define as "fixed axes," axes that are fixed relatively to the distant stars. Or, again, if a body of homogeneous fluid, assumed to be isolated, preserves a spherical form when it has no rotation relatively

to certain axes, we may call these axes "fixed." We believe these two definitions to be equivalent to one another; but in practice we always refer to distant stars as a criterion of rotation in its dynamical sense.

(6) When, in the sequel, we speak of "*the velocity, or acceleration, of any point relatively to any origin,*" fixed axes through the origin will be understood. Without these axes the expression italicised would be meaningless.

[(7) Though somewhat irrelevant, it may not be without interest to enquire here the significance of the statement that "the earth rotates round the sun, and not the sun round the earth."

Kinematically the statement has no meaning; as is seen by taking earth and sun in turn as origin;—and why should we not?

Dynamically the statement *has* significance. The relative motions of sun and earth will come best under the rule of dynamical law if we refer to their centre of mass (which lies within the sun's surface) as origin, and regard the sun's centre as describing a small ellipse, the earth's centre a large ellipse, round this point. More significance than this there could not be, unless we can attribute meaning to the expression "a point fixed in space." See VI. (1).]

IV. The Experimental Nature of Dynamics.

(1) It is possible to build up such a system of dynamics that the results arrived at *must* be true; that "Dynamics," in fact, may become a branch of mathematics. Sooner or later, however, when the science is applied to realities, there will arise the question whether or no the forces and masses of this logical dynamical system work in with other and equally important conceptions of them; and we come back to experiment after all. [See II. (1) and (2).]

(2) It seems best, therefore, to start with the conceptions of, and measures of, forces and masses discussed in II., and to examine by means of direct and indirect experiment the motions produced.

(3) Other questions also arise, to be settled by experimental evidence. Is there any simple relation between the masses of bodies and their gravitational attraction for one another? Is either "mass" or "weight" affected by the shape or physical condition of a body? Has gravitational attraction a finite velocity of propagation? Can we, in dynamics, assign a meaning to "constancy in direction of velocity"?

(4) *The "Laws" of Dynamics.* Thus in dynamics as in physics we must regard our laws as based upon experiment. As in physics, they are "Laws" only in the sense that they

account in a simple manner for all phenomena hitherto observed, and enable us to predict other phenomena, so that they are themselves capable of being tested by experiment.

Our faith in them is justified, not by rigid mathematical demonstration, but by cumulative evidence direct or indirect.

If Uranus "go wrong," we look for, and we find, Neptune; and we have so much more evidence in favour of our laws.

V. Preliminary, rough, Experiments. Newton's Laws. Mass and Weight.

(1) It is by experiments performed on the surface of the earth that we are guided towards the general laws of dynamics.

We start with forces and masses, conceived of and measured as explained in II., (1) and (2); we refer velocities, accelerations and displacements to the surface of the Earth and to landmarks on it. Such experiments are rough, but they give us valuable results.

(2) If we deal at first with one body (free to move) at a time, we arrive at the conclusion that . . . (writing "mass," "force," "acceleration," for the measures of these quantities) . . .

(mass) × (acceleration) ∝ (force applied), and the acceleration takes place in the direction of the force.

(3) If now, still dealing with one mass (free to move), we apply several forces to it, we conclude that . . .

(mass) × (acceleration) ∝ (statical resultant of the forces), and the acceleration takes place in the direction of this resultant.

(4) If now we have two masses free to move, and cause a stress to be exerted between them, as (*e.g.*) by means of a light stretched elastic cord, we conclude that . . .

the mass-accelerations are equal, and opposite in sign; the centre of mass does not move.

(5) *Mass and weight.* We recognise in matter two dynamical properties; viz. *inertia*, or passive resistance to change of motion, measured by "*mass*"; and *mutual attraction* between two bodies of matter. When one body is the earth, and the other is a relatively small body on or near to the earth's surface, we call the one side of this stress, or the earth's pull on the body, the "*weight*" of the body.

For some reason, unknown to us, *weight* and *mass* appear to be proportional to one another. Also, for some reason unknown to us, both appear to be independent of the shape and physical condition of any given piece of matter.

[It seems strange, to the writer, that there are to be found those who object to the use of *two* words, *mass* and *weight*, to designate two such entirely different properties.]

(6) Newton's laws being once established, it has seemed better to start with the fundamental units of time, mass and length, and to derive the unit of force. In the engineers' system, however, we virtually start with the units of time, length, and force (the latter being the earth's pull on a standard lump of matter at London); and derive the unit of mass.

As to the unit of time, we derive it from the earth's rotation relatively to the fixed stars. If in the lapse of years or centuries our laws appear to be changing, we ascribe this to a change in the earth's rate of rotation rather than to a change in the masses of sun or earth, or to a change in the attractive stress between them. Other standards of time have been suggested; but it would be out of place to discuss them here.

The view that the writer would lay stress on is that *we attempt to frame laws that shall make everything "hang together." Being essentially experimental laws, they cannot be proved with mathematical rigidity; they are but working hypotheses.*

VI. More complete statement of the Laws of Dynamics.

(1) Starting with the results suggested by experiments of the nature indicated above, we proceed to consider more carefully the question of the origin and axes to which, in any problem, velocities, etc., should be referred.

The conclusion arrived at is that Newton's laws hold good if we:—

(i) take as "fixed" axes those which are fixed relatively to the distant stars, and judge of change of direction of velocity by these;

(ii) choose as origin either the centre of mass (*c.m.*) of the bodies between which the reactions considered take place, or any origin with respect to which [see III., (6)] this *c.m.* has uniform velocity. For reasons given in VIII., (3), and X., (3) and (7), it appears to be better to choose as origin the *c.m.* of the reacting bodies when we are examining only the reaction between them.

If, then, we use "acceleration" in the more general (or vectorial) sense, we find that Newton's laws account for such motions as those of the solar system.

The writer would emphasise the view that *our laws of dynamics deal essentially with the system of bodies between members of which the actions considered take place, with an origin as specified above (preferably their centre of mass), and with fixed axes;* and not with "isolated bodies" and "forces acting on them."

(2) *Example.* Consider the case of two bodies of masses M and m respectively, there being between them a stress F that changes when the distance changes. And let their acceleration,

velocities, and displacements, all referred to their *c.m.* and to "fixed" axes be A and a, V and v, S and s, respectively.

Then our laws of dynamics—(whose principles are here being stated, not proved)—assert that . . . .

(i) *the mass-accelerations are equal and opposite, and each measures F.*

(ii) *the changes in momenta of the two are equal and opposite, and each measures the time-integral* $\int_{t_1}^{t_2} F.dt$ *during the period considered.*

(iii) *the changes in* $\frac{1}{2}MV^2$ *and* $\frac{1}{2}mv^2$ *measure the corresponding space-integrals* $\int Fds$, *or work done; the displacements of M and m respectively along the line of the stress being measured from the c.m.*

(iv) *From* (i) *and* (ii) *it follows that the mutual stress between M and m does not affect the acceleration or velocity of their c.m., relatively to any other origin and fixed axes.*

(3) *A still simpler example.* Let us consider the two bodies above to be a smooth, un-rifled gun, of mass M, loaded with a charge of measurable potential energy, but of negligible mass, and a shot of mass m. And for simplicity let us assume that the whole has no rotation relatively to "fixed" axes [see II., (5)], and that the charge when ignited burns at such a rate that the stress exerted has a constant value F.

Referring to the centre of mass of M and m as origin, we may say that our laws of dynamics assert that (if we disregard *signs* here, since there can arise no ambiguity of meaning) . . . .

$$MA = F = ma \qquad \text{(i)}$$

$$MV = F.t = mv \qquad \text{(ii)}$$

$$F.S = \tfrac{1}{2}MV^2 \qquad \text{(iii)}$$

$$F.s = \tfrac{1}{2}mv^2. \qquad \text{(iv)}$$

whence

$$\frac{V}{v} = \frac{m}{M} = \frac{S}{s}, \qquad \text{(v)}$$

where V and v are the final velocities, and S and s the final displacements of the masses M and m respectively, relatively to their *c.m.*, during the action, and t is the time of action (duration). Evidently also, $F(S+s) = \frac{1}{2}MV^2 + \frac{1}{2}mv^2 =$ *total work done by the charge on gun and shot together.* ....(vi)

VII. The work done by the gases.

(1) It is in considering *work* and *kinetic energy* that errors, due to neglect of definiteness as to the origin to be chosen, most

frequently arise. So we will consider this matter further here and in VIII., IX., and X. All will agree that $F\times(S+s)$ in (vi) above measures the total work done by the charge on gun and shot, and that consequently $(\frac{1}{2}MV^2+\frac{1}{2}mv^2)$ will rightly be called the gain in K.E. of the system.

(2) Though the gain in K.E. of the system, and the total work done, are the main point, still it is not without interest to enquire in what real sense we may call $F.s$ the work done on the shot, $F.S$ the work done on the gun, $\frac{1}{2}mv^2$ the K.E. of the shot, and $\frac{1}{2}MV^2$ the K.E. of the gun, in the system. But in making this enquiry it must not be forgotten that our dynamics deals essentially with a system; and that "the K.E. of an isolated body" is as meaningless as "the absolute velocity of a body," and "the work done by a force acting through a certain distance" is as meaningless as "absolute displacement." [See Maxwell's *Matter and Motion*, § CIX.]

(3) If gun and shot *came to rest again relatively to their c.m.* by striking and being stopped by sandbanks of relatively infinite mass which are at rest relatively to the *c.m.* of gun and shot, it will be admitted that quantities of heat measured by $\frac{1}{2}MV^2$ and $\frac{1}{2}mv^2$ respectively will be given out. Hence in the system of gun and shot, the masses M and m may reasonably be said to have these respective shares of the total K.E. in equation (vi) due to the explosion.

(4) Or again, not to introduce these infinite masses which are not concerned in the reaction considered, let gun and shot come to rest relatively to their *c.m.* owing to an attractive stress between them, and then return (under this attractive stress) towards one another. When in the initial relative position let there be inelastic collision, a non-conducting mat being interposed. It will be admitted, the writer thinks, that the heats given out on the two sides of the mat will again be $\frac{1}{2}MV^2$ and $\frac{1}{2}mv^2$, respectively, as before; the sudden transition from motion relatively to the *c.m.* to rest with respect to it, being virtually the same in nature as in (3) above.

Here again we see that the masses M and m can reasonably be said to share the total energy of the explosion in the proportion $F.S$ and $F.s$, or $\frac{1}{2}MV^2$ and $\frac{1}{2}mv^2$, *i.e.* in the inverse ratio of the masses.

(5) So again if the two masses M and m oscillate with respect to one another under the action of a (massless) spring of perfect elasticity that connects them, though the total energy of the system is the one thing of which we can speak without ambiguity, yet we may reasonably speak of the spring doing work on the masses of amounts $F.S$ and $F.s$, and of the masses doing work on the spring of amounts $F.S$ and $F.s$, alternately; measuring displacements as before from the centre of mass.

VIII. Lack of significance, or even error, due to failure to refer to the c.m. of the reacting bodies.

(1) Let us now, in the simple case of VI., (3), *refer to an origin relatively to which the c.m. of the two reacting bodies has a constant velocity V.* [See III., (6).]

We find that as regards mass-acceleration, change of momentum, and total work done by the explosion [see X., (1)], we obtain the same results as when we refer to the *c.m.* So that, as stated in VI., (1), (ii), any such origin will for most purposes do as well as the *c.m.* itself. But as regards the distribution of the energy of the explosion between gun and shot, it would seem that we obtain results capable of being interpreted reasonably [see VII., (3), (4), and (5)] when we refer to the *c.m.*, and somewhat meaningless results when we refer to an origin with respect to which the *c.m.* has uniform velocity. For it is clear that by choosing an origin for which V has a suitable value, we may have the shot appearing to retreat before the gases (*i.e.* to have work done on it), to stay still (*i.e.* to have no work done on it), or to advance on the gases (*i.e.* to have negative work done on it); the total work done on the system of gun and shot remaining the same as before. And in view of VII., (3), (4), and (5), such a view appears to be lacking in significance.

In the same way we might be led to speak of a gun of 45 tons being projected *from* a shot of 1 cwt.; of a man thrown by a stone; of steam, "kicking off from" a piston of small mass, driving the cylinder and framework of an engine and the earth to which the framework is fixed; or even of one side of a press doing work on a book, and of the book doing equal work on the other side of the press. And the discussion of the oscillating bodies of VII., (5), would suffer in significance for similar reasons.

In X. the advantage of referring to the *c.m.* as origin appears in a somewhat different shape; while in XI., (5) and (6), it is seen how such reference leads to the ordinary treatment of the cases, of very common occurrence, in which the one of two reacting bodies is of relatively infinite mass.

(2) Let us now, in the simple case of VI., (3) *refer velocities, etc., to an origin with respect to which the c.m. has an acceleration* [*see III.*, (6)].

We now have not only lack of significance but also error; all is in confusion. Let us, for example, take the gun or the shot as origin; both having acceleration relatively to the *c.m.* during the action of the stress.

Referring to the gun as origin we should get, for the motion of the shot, $F=m(A+a)$; while, referring to the shot, we should get for the motion of the gun $F=M(a+A)$; where F should be one and the same force. And for the K.E. due to the explosion we should get either $\frac{1}{2}m(V+v)^2$, or $\frac{1}{2}M(V+v)^2$; different values.

Another error, an error in estimating the total gain in K.E. during the explosion, if this occupied a finite time, is indicated in X., (6). With such an origin, then, we have actual error.

(3) We see then that if we are to obtain results that are in accordance with the laws of dynamics we must choose an origin relatively to which the *c.m.* of the system has uniform velocity; and that if we are to give a *reasonable* account of the distribution of the energy of the system between the various masses composing it, we should refer to the *c.m.* itself as origin.

Hence it is better to refer all displacements, velocities, etc., to the c.m. of the system of the bodies between which the reactions take place when discussing these reactions; though for most purposes any origin with respect to which this c.m. has uniform velocity will do equally well [*see III.*, (6)]. W. LARDEN.

(*To be continued.*)

NOTES ON THE THEORY OF THE REVERSIBLE PENDULUM. (PART II.)

[*Part I. appeared in the May issue under the title* "*On the Adjustment of Kater's Pendulum.*"]

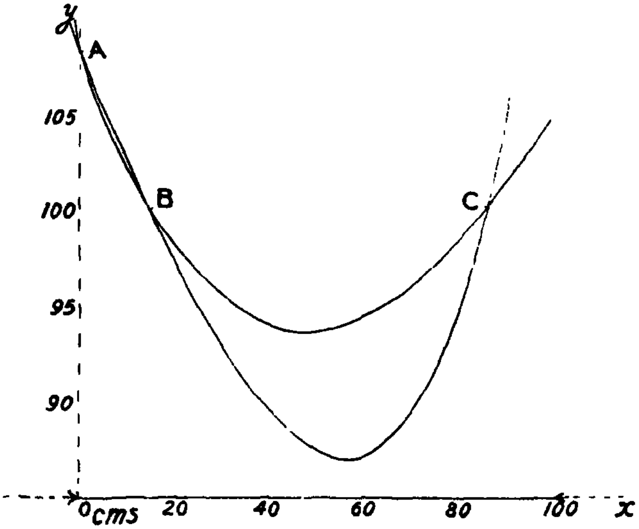
7. We may now turn our attention to the questions which arise specially in connection with a reversible pendulum, such as Kater's. Of course the object in view is so to place two knife-edges on opposite sides of the C.G. that the period shall be the same from either; the two positions being selected according to the rule at the end of § 4. But the usual procedure will be to decide on the distance between the knife-edges at the start, say one metre; then to clamp them to the more or less uniform bar of the pendulum, using a standard distance-piece to secure the proper interval between them; and finally to adjust a sliding weight in such a position as to make the period the same for both. We have seen in § 2 how the period, or rather the length of the equivalent S.P., varies with the position of the sliding weight; but it will be convenient now to change the notation. Let I_1, I_2, ... denote the moments of inertia of the various masses making up the pendulum, each about its own C.G.; M_1, M_2... the masses; x_1, x_2... the co-ordinates of the centres of gravity, referred to one of the knife-edges as origin; let the co-ordinate of the other knife-edge be d; and let the letters I, M, x without suffixes refer to a moveable mass which is being adjusted. We shall at once see how much simpler it is to discuss the graph whose ordinate is the length of the S.P., than the one where the period is used; for the position which the moveable mass should occupy of course corresponds to the intersection of the two graphs relating respectively to the two

knife-edges; and it is surely better to have to deal with the intersections of conics than of cubics. Let us take I_1, M_1, x_1 to refer to the bar and any adjuncts whose position is invariable; I, M, x will at present refer to the "heavy bob" of a Kater pendulum. The ordinate for one knife-edge is

$$\frac{(I_1+I)+M_1x_1^2+Mx^2}{M_1x_1+Mx};$$

for the other it is

$$\frac{(I_1+I)+M_1(d-x_1)^2+M(d-x)^2}{M_1(d-x_1)+M(d-x)};$$



§ 7, with data from § 10; $x_1=75$. Compare the character of the crossing at B and at C.

and the bob has to be so adjusted as to make these equal. Multiplying up, we have a cubic; so that there are three positions possible. The cubic is factorisable, so that all three can readily be found. But it may be said that two conics in general intersect in four points, and imaginaries can only come in in pairs; where then is the fourth intersection in the present case? The answer is that the two curves also cross at infinity; each has an asymptote parallel to the y-axis. The other three solutions may be most easily obtained as follows. The numerator of the second fraction may be written

$$(I_1+I)+d\{M_1(d-2x_1)+M(d-2x)\}+M_1x_1^2+Mx^2;$$

so that when the two denominators are equal, *i.e.* when

$$M_1(d-2x_1)+M(d-2x)=0$$

and the masses have equal and opposite moments about the "middle point," the two numerators are also equal; hence one solution is $x=\frac{(M+M_1)d-2M_1x_1}{2M}$. The other two are obtained by remembering that when the bob is in its right position the two knife-edges are separated by an interval equal to the length of the equivalent S.P.; *i.e.*

$$d=\frac{(I_1+I)+M_1x_1^2+Mx^2}{M_1x_1+Mx},$$

a quadratic whose roots are

$$x=\frac{d}{2}\left\{1\pm\sqrt{1-\frac{4(I_1+I+M_1x_1^2-M_1x_1d)}{Md^2}}\right\};$$

these two positions, then, are symmetrical with respect to the "middle point." (The first solution, it will be noticed, is that which places the C.G. of the whole pendulum just at the middle point.) Let the three positions be called A, B and C respectively.

8. We may now consider one or two special points that may arise. In the first place, we have equated the essentially positive quantity d to two fractions, each of which has an essentially positive numerator; we have therefore restricted ourselves to cases in which the denominators are also positive, which means that the C.G. of M and M_1 is bound to lie between the knife-edges. Hence the above method of solution effectually guards us against including those cases in which the pendulum swings the same way up from both knife-edges and at the same rate. We could only obtain these inappropriate cases by writing

$$d=\pm\frac{(I_1+I)+M_1x_1^2+Mx^2}{M_1x_1+Mx}=\mp\frac{(I_1+I)+M_1(d-x_1)^2+M(d-x)^2}{M_1(d-x_1)+M(d-x)}.$$

As a consequence, the three intersections may be shewn to be included between the asymptotes

$$x=-\frac{M_1}{M}x_1 \text{ and } x=d+\frac{M_1}{M}(d-x_1).$$

The next point relates to the special circumstances under which the two graphs may touch each other. The condition that the A intersection should coincide with either B or C is given by the equation

$$M_1x_1(d-x_1)=\frac{M(I_1+I)}{M_1+M}+(M_1-M)\frac{d^2}{4}.$$

This gives $$x_1=\frac{d}{2}\left\{1\pm\sqrt{\frac{M}{M_1}-\frac{4M}{M_1}\cdot\frac{(I_1+I)}{(M_1+M)d^2}}\right\};$$

the upper sign corresponds to coincidence of A and B at

$$x=\frac{d}{2}\left\{1-\sqrt{\frac{M_1}{M}-\frac{4M_1(I_1+I)}{M(M_1+M)d^2}}\right\},$$

the lower sign to coincidence of A and C at the same distance on the other side of the middle point. The B and C intersections coincide when $M_1x_1(d-x_1)=(I_1+I)-\frac{Md^2}{4}$, *i.e.* when

$$x_1=\frac{d}{2}\left\{1\pm\sqrt{\frac{M_1+M}{M_1}-\frac{4(I_1+I)}{M_1d^2}}\right\},$$

and it will be noticed that these limits are further apart than those for the other kind of coincidence; this case corresponds to $x=\frac{d}{2}$. If x_1 should lie outside these limits, it will be impossible ever to secure equality of period except by putting M at

$$x=\frac{(M+M_1)d-2M_1x_1}{2M};$$

or, to state the same fact in a more useful form, M must be at least as large as $\frac{4}{d^2}\{(I_1+I)-M_1x_1(d-x_1)\}$ or adjustment for equality of period will only be possible by using the A intersection, and the C.G. of the whole pendulum is then at the "middle point." We proceed to shew that this is highly undesirable.

9. If t_1 and t_2 denote the periods, and h_1 and h_2 the distances of the C.G. of the whole from the *corresponding* knife-edges, then the usual formula is

$$\frac{8\pi^2}{g}=\frac{t_1^2+t_2^2}{h_1+h_2}+\frac{t_1^2-t_2^2}{h_1-h_2}.$$

The distance h_1+h_2 is what we have called d and can be found accurately; h_1-h_2 cannot, and therefore it is sought to make the second term small. This is done by adjusting t_1 and t_2 very nearly to equality; to do so exactly would be tedious, but the experiment is continued until the difference is small. For small alterations the factor t_1+t_2 is practically constant, viz. about $4\pi\sqrt{\frac{h_1+h_2}{g}}$, and the term $\frac{8\pi^2}{g}$ is constant, so that the relative importance of the doubtful term for a given difference t_1-t_2 depends entirely on h_1-h_2, which ought to be large; in fact, the ideal to be aimed at is $h_1 \backsim h_2=d$, *i.e.* the case of a simple pendulum, as we might have expected. In particular, any adjustment which gives $h_1=h_2$ and consequently $t_1=t_2$, is bad; for the second term is of the indeterminate form $\frac{0}{0}$, and may easily be shewn to have a finite value, viz.

$$\frac{4\pi^2}{g}\left\{1-\frac{(2x_1-d)(d-2x)}{d^2}-\frac{4(I+I_1)}{(M+M_1)d^2}\right\}.$$

Thus in cases which nearly give coincidence of A with B or C, only that intersection must be used with which A does not nearly coincide; cases which nearly cause B and C to coincide are only objectionable when the centre of gravity of M_1 is near the middle point. To sum up the whole matter, we may say that the adjustment is better in proportion as the intersection which is used is quite distinct from the other two.

10. Before proceding further, we may examine some numerical examples, based on a pendulum of the usual pattern and dimensions. The bar is approximately uniform; its length is about 150 cms., its mass about 2000 gms., and its moment of inertia with respect to its C.G. about 5×10^6 gm. cm^2. The attached mass weighs about 1000 gms., and as its moment of inertia about its own C.G. is only about $\frac{1}{4}$ per cent. of that of the bar we will consider it to be a particle. Let us begin with the knife-edges so placed that $x_1 = 50$ cms.; we shall suppose them to be always 100 cms. apart. The three values for x are then (A) 50, (B and C) 0 and 100; rejecting A, we see that for B or C $h_1 \backsim h_2 = \frac{100}{3}$ cms. If we move the knife-edges so that $x_1 = 75$ cms., the maximum value possible, then B and C are at 14·65 and 85·35, and

$$h_1 \backsim h_2 = \frac{100}{3}\left(1 - \frac{1}{\sqrt{2}}\right) \text{ or } \frac{100}{3}\left(1 + \frac{1}{\sqrt{2}}\right)$$

when M is at B or C respectively. Comparing these with each other and with the former value, we see at once the advantage of asymmetry; an error in fixing the position of the C.G. of the whole would be nearly 34 times as serious with M at B as if M were at C. For the percentage error in g varies inversely as the square of $h_1 \backsim h_2$; doubling $h_1 \backsim h_2$ would not merely halve the relative importance of a given error in $h_1 \backsim h_2$ but would also halve the importance of the term $\frac{t_1^2 - t_2^2}{h_1 - h_2}$. Suppose we continue to adjust t_1 and t_2 till they differ only by τ, each being about 2 secs.; and suppose we make an error δ in locating the C.G.; this means a fractional error $\frac{2\delta}{h_1 - h_2}$ in the denominator, or $\frac{2\delta}{(h_1 - h_2)} \frac{(t_1^2 - t_2^2)}{(h_1 - h_2)} \cdot \frac{(h_1 + h_2)}{(t_1^2 + t_2^2)}$ in g; this is approximately $\frac{2\delta \cdot \tau \cdot 100}{(h_1 - h_2)^2 \cdot 2}$, *i.e.* $\frac{100\tau\delta}{(h_1 - h_2)^2}$ or $\frac{10000\tau\delta}{(h_1 - h_2)^2}$ per cent.

In an experiment purporting to give g correctly to 1 in 10000, this fraction should be less than ·01; suppose τ to be 0·01 sec., and δ to be 0·1 cm., then $h_1 \backsim h_2$ ought to be at least 32 cms. If the uncertainty in the position of the C.G. were greater, we must either make the periods more nearly equal, or use a more unsymmetrical arrangement; the latter would involve less trouble. With $x_1=50$ and $x=0$ or 100 the above values of τ and δ would give about $\cdot01\,{}^{\circ}/_{\circ}$; with $x_1=75$ and $x=85{\cdot}35$, about $\cdot003\,{}^{\circ}/_{\circ}$; with $x_1=75$ and $x=14{\cdot}65$, about $\cdot105\,{}^{\circ}/_{\circ}$; the difference in the latter cases being very noteworthy. And if we take the A intersection given by $x_1=50$, $x=50$, and imagine the periods precisely equal, then the slightest mistake in h_1-h_2 would make the term $\frac{t_1{}^2-t_2{}^2}{h_1-h_2}$ zero instead of $\frac{0}{0}$, *i.e.* $\frac{4\pi^2}{3g}$, or $\frac{1}{6}$ of the whole!

Or again, suppose $x_1=50$ and $x=50{\cdot}3$, so that $h_1=50{\cdot}1$ and $h_2=49{\cdot}9$. Then if h_1 were wrongly estimated as $50{\cdot}2$, we should make the term $\frac{t_1{}^2-t_2{}^2}{h_1-h_2}$ only half as large as it should be, and yet it is about $\frac{1}{6}$ of the total; or if the mistake were in the other direction, we should actually make the term infinite instead of $\frac{4\pi^2}{3g}$. These illustrations shew how necessary it is to avoid having the C.G. about midway between the knife-edges.

In the pendulum at present under discussion, although the bar is 150 cms. long, certain fixed attachments limit the range over which the knife-edges can be moved, so that the value of x_1 cannot conveniently exceed 60 cms. This gives $x=2$ or 98 approx.; choosing the value 98, $h_1-h_2=45{\cdot}3$ cms., which with $\tau=0{\cdot}01$ sec. and $\delta=0{\cdot}1$ cm. easily ensures an accuracy of $0{\cdot}01\,{}^{\circ}/_{\circ}$.

The A and B intersections coincide when $x_1=70{\cdot}41$ and $x=9{\cdot}18$; consequently it would be very important to choose the alternative value for x, viz. $90{\cdot}82$. Similarly when $x_1=29{\cdot}59$ and $x=90{\cdot}82$ the A and C intersections coincide, so that here we should select $x=9{\cdot}18$. The B and C intersections would only coincide for $x_1=85{\cdot}35$ or $14{\cdot}65$, and it is impossible in the present instance for x_1 to fall outside the limits 25 and 75, so that we have no concern with this case.

11. The connection between the position of the centre of gravity of M_1 and the corresponding positions of M which give equality of period may be instructively shewn in the following graphical way. The co-ordinates x_1 and x of M_1 and M respectively are connected by the equation at the end of § 7, viz.

$$M_1x_1(d-x_1)+Mx(d-x)=I_1+I.$$

If then we plot values of x_1 and x as abscissae and ordinates respectively, it is clear that we get an ellipse with its centre at $\left(\frac{d}{2}, \frac{d}{2}\right)$ and of semi-axes

$$\frac{1}{2}\sqrt{\frac{(M_1+M)d^2-4(I_1+I)}{M_1}},\ \frac{1}{2}\sqrt{\frac{(M_1+M)d^2-4(I_1+I)}{M}};$$

in the special case when the attached mass M equals the mass of the remainder M_1, this ellipse becomes a circle. This illustrates the point that for any given value of x_1 there are two values of x symmetrical with respect to $\frac{d}{2}$, and *vice versa.* It also shews that, as the ellipse contracts to a point when

$$(M_1+M)d^2=4(I_1+I),$$

the smallest mass which will render adjustment possible is

$$\frac{4(I_1+I)}{d^2}-M_1.$$

We see, further, that when the centre of gravity of M_1 is half-way between the knife-edges, M is as near one of them as it ever will be: while if the centre of gravity of M_1 is further from the middle than $\frac{1}{2}\sqrt{\frac{(M_1+M)d^2-4(I_1+I)}{M_1}}$, M is too small to effect adjustment even when placed at the middle point.

If instead of x we take h_1, *i.e.* $\frac{M_1x_1+Mx}{M_1+M}$, for ordinate and so trace the connection between the position of the centres of gravity of M_1 and M_1+M, we obtain another ellipse,

$$(M+M_1)h_1^2-2M_1h_1x_1+M_1x_1^2=Mdh_1-M\cdot\frac{I_1+I}{M_1+M}.$$

Its axes are inclined, but it lies within the bounds

$$x_1=\frac{d}{2}\left\{1\pm\sqrt{\frac{M_1+M}{M_1}-\frac{4(I_1+I)}{M_1d^2}}\right\},\quad h_1=\frac{d}{2}\left\{1\pm\sqrt{1-\frac{4(I_1+I)}{(M_1+M)d^2}}\right\}.$$

At the place where it touches the latter lines it cuts the original ellipse; the centre of gravity of the whole is then as far from the middle point as it ever can be with the given values of M and M_1. Its intersections with the line $h_1=0$ indicate the values of x_1 for which A coincides with B or C; see § 9.

12. Let us proceed with the numerical illustration of § 10, and continue the discussion of the case in which x_1 was 50 and x was 100. Now in the first place it is not convenient to construct M so that it can be fixed at or near a knife-edge; it cannot be put nearer than 90 on the one side or 110 on the other; it will be remembered that the knife-edge is at 100. The desire for asymmetry leads us to choose 110; having fixed it there, let us replace the symbols I, M, and x by I_2, M_2 and x_2. We have now destroyed the equality of period and to restore it we must add another mass; let us take I, M, and x as referring to it. As a matter of fact, the pendulum has two other masses, one resembling the heavy bob but smaller, the other a screw arrangement for fine adjustment; however, let us consider these as one

for the present. The equation will now be

$$I_1+I_2+I=M_1x_1(d-x_1)+M_2x_2(d-x_2)+Mx(d-x);$$

and it may be as well to remark here that the general equation is $\Sigma(I)=\Sigma\{Mx(d-x)\}$. In our case,

$$5\times10^6=(2000 . 50 . 50)-(1000 . 110 . 10)+Mx(d-x);$$

and since the maximum value of $x(d-x)$ is $\frac{d^2}{4}$, *i.e.* 2500, we see that M must be at least 440 gms. But as M includes two portions, and as the effect of the two (wherever placed) is less than if they were massed together at the middle point, we shall really require more than 440 gms., say 480 gms.; then if it were a single particle its position would be given by

$$480x(d-x)=11\times10^5.$$

But it consists of a light bob (say 220 gms.) at x, and a screw attachment (260 gms.) at y; then

$$220x(d-x)+260y(d-y)=11\times10^5,$$

the usual elliptical law. We see that as the semi-axes of this ellipse are 21·32 and 19·61, x must be between 71·32 and 28·68, and y between 69·61 and 30·39. In selecting a suitable pair of values we of course try to increase the asymmetry, but we must also bear in mind that for constructional reasons the two centres of gravity must be at least 12 cms. apart. Suppose we place the C.G. of the screw attachment at 69, and that of the small bob at 55·3; this will satisfy all the conditions, and will at the same time keep the screw well away from the position 50, where it would be very ineffective for purposes of adjustment. Lastly let us consider what effect we shall produce on the two periods by moving this screw-portion 1 cm. The denominator

$$(2000\times50)+(220\times55{\cdot}3)+(260\times69)+(1000\times110)=240000 \text{ q.p.},$$

hence the numerator is 24×10^6; to move the 260 to 70 will increase the denominator by 260 in 24×10^4; it will increase the numerator by $260\times(70^2-69^2)$, *i.e.* by 260×139 in 24×10^6; the length of the simple equivalent pendulum is therefore increased by 260×39 in 24×10^6, and the time of swing by 130×39 in 24×10^6, say by 21 in 10^5. The time of swing from the other knife-edge may be shewn to be increased by 47 in 10^5; thus a movement of 1 cm. will only be enough to compensate an inequality in the period of about ·0005 sec. And the screw only moves about 120 of the 260 gms., so that this ·0005 sec. really corresponds to a movement of about 2·2 cms. It is a common mistake to waste time with this fine adjustment when the inequality of period is really quite out of its range and ought to be dealt with either by moving the attachment bodily over several cms. or even by shifting the heavy bob.

13. Of course the time of swing of the Kater Pendulum is found in terms of that of a standard clock by the method of "coincidences"; and it is to be hoped that very few laboratories employ the troublesome "eye and ear method" of judging coincidences, which almost demands that the observer should possess a sixth sense. A far better plan is to arrange that the standard pendulum and the Kater pendulum shall each carry a vertical slit; a beam of light sent through the clock-case from back to front (by means of a mirror, say) can only reach the observer's eye through these slits when they are in line with a third slit so fixed behind them both as to be in line with them when both pendulums hang at rest. A flash of light is then the signal denoting a "coincidence"; naturally there will be several consecutive swings which will give flashes, but the observer notes down all and selects the middle one as indicating the instant of exact coincidence.

The experiment is one that can easily be performed without help; the observer begins counting with the seconds' hand of the clock, transfers his counting to the Kater pendulum, *continues counting as he watches through his telescope until he sees* a flash, writes down (without losing count) the figure counted at that moment, and when the flashes cease he also writes down the figure corresponding to the last one; still without losing count, he now refers to the seconds' dial once more, and if he finds he is one second out (as is most likely) he corrects his figures accordingly, and has nothing more to do till the next period of flashing. With knife-edges fixed a metre apart this will be in about 5 minutes, a very convenient interval.

Summary. §§ 1–3 shew, by calculation and graphically, how the period of a compound pendulum will be affected by shifting part of its mass or by attaching additional mass; §§ 4 and 5 apply similar graphical methods to a familiar laboratory experiment on the pendulum; § 6 refers to a less convenient graphical method. In § 7 it is shewn that in the case of a reversible pendulum equality of period can (subject to certain restrictions) be secured by attaching a given mass in one or other of three positions, corresponding to the three intersections of two graphs; in § 8 the restrictions are dealt with, while § 9 shews that one (or in certain cases two) of the three positions must be rejected as unsuitable; numerical illustrations are added in § 10. It is proved in § 11 that shifting one part of the mass of the pendulum can (within limits) be compensated by shifting another part, provided the two movements are connected by a relation of an elliptical character, and this receives illustration in § 12. A convenient method of carrying out pendulum experiments is briefly referred to in the concluding paragraph.

A. O. Allen.

REVIEW.

Text-Book of Light. 4th Edition. Univ. Tut. Press.

This book has been thoroughly revised and very considerably enlarged. The experimental side of the subject receives attention, and the clear practical directions for performing experiments which are given greatly enhance the value of the work. A chapter is devoted to the experimental determination of the constants of Lenses, etc. Dispersion is fully treated in a simple manner; the chapter on it includes a good discussion of ordinary spectra, some remarks on Absorption Spectra, Colour, Rainbows, etc., and forms a notable section in the book. The chapter on Velocity of Light is improved by an account of the Corpuscular and Wave Theories. The number of problems given has been increased; and the student who works through them will have ample exercise in this important branch of optics. The get-up of the work is in the main highly satisfactory; the figures are well and clearly executed. Here and there one finds an overcrowded page; this however detracts but little from the general excellence of the volume. The book, as is stated in the preface, is a very full treatment of those parts of geometrical optics which do not require advanced Mathematics, and it is, indeed, difficult to imagine a better. T. R. DINGAD DAVIES.

MATHEMATICAL NOTES.

212. [D. 6. b.] *The Exponential Function.*

How to best introduce the learner to the exponential function is a difficulty which, it seems to me, needs to be discussed.

I take it that hitherto the first appearance of this function has been in Algebra. After the binomial theorem for positive integral indices, came a short discussion of convergence, and then, in rapid succession, the general binomial series, the exponential series, and the logarithmic series.

In this order the extension of the binomial theorem seems natural to the learner, and as the exponential theorem leads rapidly to that by which logarithms can be calculated, its introduction probably appears natural also.

In future, however, we are to cater for army candidates and the would-be engineer, as well as for those about to read for the mathematical tripos: one result of this is that the exponential function will first appear in the differential calculus, and will form the most serious difficulty in the earlier part of that subject.

In attacking this difficulty it seems to me that, first, the importance of a function y such that $\frac{dy}{dx}=ky$ should be impressed on the learner by examples such as: increase of population; compound interest; diminution of light in passing through several consecutive panes of glass; increase of resistance when a rope is coiled round a post; cooling of a hot body; etc.

After this several methods may be adopted.

(α) Start fresh, and try to differentiate $y=a^x$;

$$\text{here } dy=a^x(a^{dx}-1).$$

If we assume $a^{dx}-1$ to be a multiple of dx, say Adx, then A will be a function of a; let e be the value of a that makes $A=1$; the e is defined by $d.e^x=e^x.dx$. If now we assume that e^x can be expressed as a series of powers of x, we deduce $e^x=1+x+\frac{x^2}{\underline{|2}}+\ldots\equiv\exp(x)$, say.

Here by a series of assumptions we have proved, or rather avoided the necessity of proof of the 'exponential theorem' that $\{\exp 1\}^x=\exp x$.

(β) Again, we may lay stress on compound interest when the interest-period is small, and consider $\underset{n=\infty}{\mathrm{L}}\left(1+\frac{x}{n}\right)^n$.

By using the binomial theorem, and by burking the question of approaching the limit except near the beginning of the series, we can easily obtain a sort of proof that $\underset{n=\infty}{\mathrm{L}}\left(1+\frac{x}{n}\right)^n=\exp x$, and $\underset{n=\infty}{\mathrm{L}}\left(1+\frac{1}{n}\right)^n=\exp 1.$

Here again we have avoided the proof of the exponential theorem.

(γ) On the other hand, it seems more straightforward to proceed directly from $\frac{dy}{dx}=ky$. If we assume for y a power series in terms of x, we obtain $y=A\exp(kx)$, where A is a constant; and in particular if $k=1$ and A be dropped, $y=\exp(x)$.
Unfortunately by this method we have not proved that $\exp x=(\exp 1)^x$.

Personally I should like to adopt (γ). Having seen the importance of the solution of $\frac{dy}{dx}=ky$, and having determined its form, $y=\exp x$, I should like to discuss the multiplication $\exp x_1 . \exp x_2$, and prove this product to equal $\exp(x_1+x_2)$, in fact to prove the exponential theorem in the most satisfactory way.

To give a rigorous proof by this method is certainly easier than to make (β) rigorous, but it would be interesting to collect opinions on the merits of (α), and to see whether a satisfactory proof of the exponential theorem can be given by that method.

C. O. Tuckey.

213. [K. 2. c.] *The nine-point circle.*

It is well known that if O is the orthocentre of a $\triangle ABC$, then A is the orthocentre of $\triangle OBC$; and so on.

The symmetry that thus exists between the four points, leads to many interesting properties, especially in connection with the nine-point circle. In stating these properties, however, the symmetry is not often mentioned, so that the following account of well-known theorems may perhaps interest some teachers.

Let O be the orthocentre of $\triangle ABC$, then $OABC$ may be regarded as a tetrahedron in one plane, with 6 edges.

The 4 points form 3 $\triangle$s with a common pedal $\triangle XYZ$ (X on BC, etc.).

The 6 mid-points of the edges (D, E, F for BC, etc., P, Q, R for AO, etc.), form 3 rectangles (such as $PQDE$) whose sides are parallel to and half of the edges.

Each pair of rectangles has a common diagonal, and the 3 diagonals of the rectangles meet at a point (N say) which is the centre of a ⊙ round DEF, PQR and passing through XYZ (since $\angle DXP$ is right, etc.).

This circle has radius half the circum-radius for either ABC, OBC, etc. since in each case it is the circum-circle of a similar $\triangle$ of half the linear dimensions.

The point N is the c.g. of equal particles at $AOBC$ (which shows that PD, QE, RF meet in a point) and therefore divides in ratio 3 to 1 the joins of O to c.g. of $\triangle ABC$, A to c.g. of $\triangle OBC$, etc.

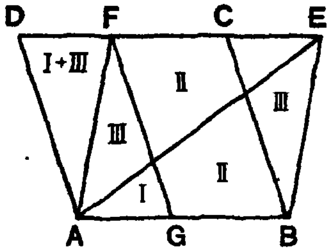
The point N also bisects the join of O to the circum-centre of $\triangle ABC$ (being centre of ⊙ through DX, FZ) and the join of A to circum-centre of OBC, etc.

C. O. Tuckey.

214. [V.] *An illustration by dissection.*

The following is an illustration by dissection of the theorem: "If a parallelogram $ABCD$ and a triangle ABE be on the same base AB and between the same parallels AB, DCE, the parallelogram shall be double of the triangle," for the case when the apex of the triangle falls outside the parallelogram and CE is not greater than AB.

AF in the figure is drawn parallel to BE and FG parallel to CB. The triangle ABE is now divided into three portions I., II., III., which can be made to occupy one-half of the parallelogram $ABCD$ as indicated by the figure.

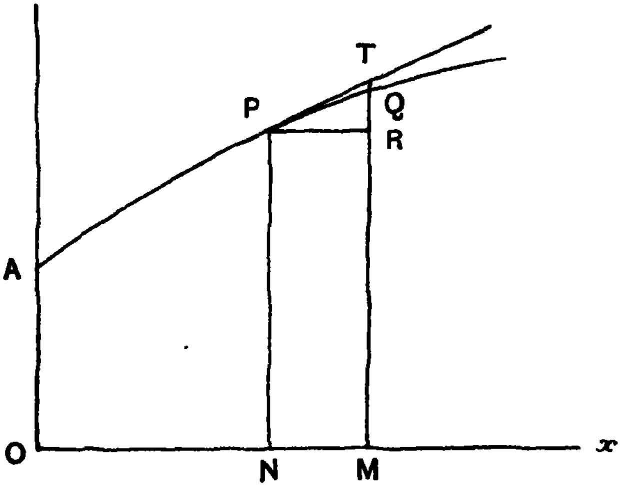


A. E. Pierpoint.

La Martinière College, Lucknow.

215. [V. 4. c.] *A method of obtaining a second approximation.*

In following the practice, nowadays common, of teaching the early parts of the Differential and Integral Calculus simultaneously, geometrical illustrations are useful and an early application of the calculus in making approximations usually evokes the interest of the student. This note gives a simple method of getting a second approximation by a geometrical illustration.



$f(x)$ denotes the area between a curve, an ordinate, and the axes. After showing, as is usual, that $f'(x)=y$ and pointing out that $\frac{dy}{dx}=f''(x)$, a glance at the figure shows that if Q is near P and NM is Δx, a first approximation to the area $AQMO$ is $APNO$+the rectangle $PRMN$, *i.e.*

$$f(x+\Delta x)=f(x)+\Delta x f'(x).$$

A second approximation is $APNO+PRMN+\triangle PTR$.

The $$\triangle PTR = \tfrac{1}{2}PR\,.\,TR = \tfrac{1}{2}PR^2 \tan TPR$$
$$= \tfrac{1}{2}(\Delta x)^2 \frac{dy}{dx} = \tfrac{1}{2}(\Delta x)^2 f''(x)\,;$$
$$\therefore\ f(x+\Delta x) = \text{approximately } f(x) + \Delta x f'(x) + \tfrac{1}{2}(\Delta x)^2 f''(x).$$

The case of a negative $\frac{dy}{dx}$ is a useful exercise for a pupil.

W. M. Roberts.

216. [L_1. 15. f.] Mr. Trachtenberg's cubic is not *new*, for, as was once sardonically observed, it perpetually haunted the evening meetings of the M.A. Some time ago I drew up a paper on "the locus of a point at which two given conterminous straight lines subtend equal or supplementary angles." The locus was found to be a strophoid, *i.e.* the inverse of a rectangular hyperbola, and for the sake of simplicity is most naturally approached without consideration of the conic at all. Even at school one comes across it in the question: If A, B are fixed points external to a given circle, find a point P on the circumference such that $AP+PB$=a minimum. The proof by inversion is quite simple:

If $O\hat{P}A = O\hat{P}B$, then $O\hat{a}p = O\hat{b}p$ or $O\hat{a}b + p\hat{a}b = O\hat{b}a + p\hat{b}a$;
$$\therefore\ p\hat{a}b \sim p\hat{b}a = O\hat{a}b \sim O\hat{b}a,$$
or the locus of a p is a rect.-hyp. having its centre at the middle pt. of ab, and passing through O, a, b, etc.

If P be a point on any one of a series of confocal conics, the normal at which passes through a fixed point O, then SO, $S'O$ subtend equal angles at P. Mr. Trachtenberg notices this, but does not bring it out clearly that the locus of P is independent of all considerations of tangents and normals to conics.

R. F. Davis.

217. [L_1. 7. d; 14. a.] Professor Ramaswami Aiyar's researches (in the March No. of *Ed. Times*) on the foci of an inscribed conic are, I think, worthy of attention.

He has shown that:—(see *Edinburgh Mathematical Society's Proceedings*, vol. xv., p. 74)

If P, Q are isogonal conjugate points with respect to the $\triangle ABC$ (*i.e.* possible foci of an inscribed conic, $\alpha_1.\alpha_2 = \beta_1\beta_2 = \gamma_1\gamma_2$), M the middle pt. of PQ, O the circumcentre, H the orthocentre, N the ninepoints centre, then
$$OP\,.\,OQ = 2R\,.\,MN.$$
(*Note.* When P, Q coalesce in I, then $OI^2 = 2R\,.\,NI$ or $NI = \frac{OI^2}{2R} = \frac{R}{2} - r.$)

What is the GEOMETRICAL interpretation of these?

218. [L_1. 7. d; 14. a.] Wanted, a simple proof (geometrical) that the polar circle of a triangle cuts orthogonally the director circle of any inscribed conic —a proof which does not approach the triangle by way of the quadrilateral.

R. F. Davis.

219. [V. 9.] In reading the second volume of *A History of European Thought in the Nineteenth Century*, by Dr. Merz of Newcastle, published by Messrs. Blackwood a little over a year ago, and reviewed in *Nature*, January, 1905, by Professor Bryan, I was much struck with the exceptional value of Chapter XIII. on "The History of Mathematical Thought." I approached the publishers to see whether this Chapter could not be obtained separately and interleaved. They have very kindly acceded to my request, after consultation with the author; and you may possibly by this time have received a copy for review. I think this Chapter is of particular interest members of our Association, and especially to mathematical teachers; moreover, the form of publication with the additional blank pages will enable anyone to add additional notes and references about the most recent work.

It is necessary continually to remind the new members of the value of history in mathematical teaching; the back numbers of the *Gazette* have some very happy papers on this matter.

T. J. Garstang.

220. [K. 10. e.] *To draw from any two given points A, B two straight lines to a point R on a given circle, having centre at C, so that they may be equally inclined to CR.*

The following solution has the advantage that the curve made use of is independent of the data, and consequently, when once described, serves for every case that may arise.

Let $CB = a$, $CA = b$, radius of circle $= c$, $\angle ACB = 2a$.

Take for the initial vector CH, the bisector of $\angle ACB$.

Then, if a point R move so that AR and BR are always equally inclined to CR, the polar equation of the locus is

$$\frac{r}{2ab}\{(a-b)\sin a \operatorname{cosec}\theta + (a+b)\cos a \sec\theta\} = 1, \quad \ldots\ldots\ldots\ldots(1)$$

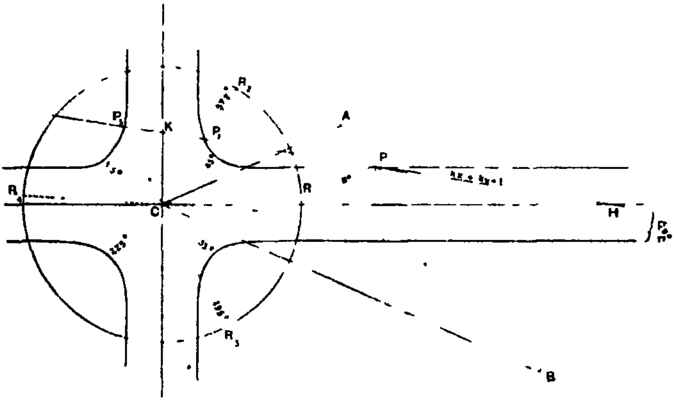
regarding CA as being on the positive side of CH.

The question becomes, what is θ when $r = c$?

Let $\qquad c(a-b)\sin a/2ab = h, \quad c(a+b)\cos a/2ab = k,$

then

$$h \operatorname{cosec}\theta + k\sec\theta = 1. \quad \ldots\ldots\ldots\ldots(2)$$



Now it is obvious that, l being some convenient length, the point $(l\operatorname{cosec}\theta,\ l\sec\theta)$ is the intersection of the straight line

$$hx + ky = l, \quad \ldots\ldots\ldots\ldots(3)$$

and the curve

$$\left(\frac{l}{x}\right)^2 + \left(\frac{l}{y}\right)^2 = 1. \quad \ldots\ldots\ldots\ldots(4)$$

This curve is easy to plot, because its polar equation is

$$r = 2l \operatorname{cosec} 2\theta.$$

On the given figure describe the line (3); on tracing paper describe the curve and then place it on the figure. Prick through the points P_1, P_2, P_3, P_4 where the curve intersects the line (3). The lines drawn through C to these 4 points cut the circle at 8 points, 4 of which,

R_1, R_2, R_3, R_4 are those required. There can be no doubt which four they are when the corresponding values of θ are considered. Fo example, although P_3 in the figure falls in the 2nd quadrant, its θ, valu is 306°, showing that R_3 must be in the 4th quadrant.

By supposing the nearer of the two given points to C to be on th positive side of CH, the branch of the curve in the 3rd quadrant is o no use, for the intercepts CH and CK of the line (3) will always be positive

H. Garfeur.

221. [K. 10. e.] In the equation in Note, viz.

$$r\{(a-b)\sin\alpha\operatorname{cosec}\theta+(a+b)\cos\alpha\sec\theta\}=2ab$$

put
$$abp=c^2(a-b)\sin\alpha\,;\quad abq=c^2(a+b)\cos\alpha,\quad r=c.$$

We then have
$$\frac{1}{2}\left(\frac{p}{\sin\theta}+\frac{q}{\cos\theta}\right)=c.$$

Let a straight line from the origin intersect the straight lines

$$r\sin\theta=p,\quad r\cos\theta=q,$$

in F and G. Bisect FG in P.

The circle $r=c$ will cut the locus of P in the four points required.

If the four values of θ be θ_1, θ_2, θ_3, θ_4, then $\Sigma\theta_1$, is a multiple of π.

W. H. H. Hudson.

222. [K. 10. e.] Another solution is as follows.

Bisect AOE internally in CY, externally in CX

Let H, K be the inverse points of A, B with respect to the circle.

Bisect HK in F and let G be the reflexion of F in CY

Fit, mechanically, a line of length c between the rectangular axes CX, CY so that its direction passes through G.

CR is parallel to the direction of this line.

There are four positions of R. W. H. H. Hudson.

223. [V. 1. a.] *Higher Trigonometry.*

In Mr. Hardy's most interesting article on the proof of the theorem $e^{ix}=\cos x+i\sin x$ he recommends the proof (which he numbers (iii))

$$\text{if } y=\cos x+i\sin x,$$
$$\text{then } \frac{dy}{y}=i\,dx\,;$$
$$\text{integrating, } \log y=ix+\text{constant}.$$

It would be of great use to teachers if he would point out how much knowledge of contour integration is supposed.

Is it necessary to prove

(α) For a contour C not containing an infinity of $f(z)$

$$\int f(z)dz=0\,?$$

(β) If C contains a simple pole at $z=a$, where $f(z)=\dfrac{M}{z-a}$,

$$\int f(z)dz=2\pi iM\,?$$

C. O. Tuckey.

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